JCDCG^3 2016

The 19th Japan Conference on Discrete and Computational Geometry, Graphs, and Games
September 2 - 4, 2016
Tokyo University of Science

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Ruy Fabila-Monroy	(Cinvestav, México)
János Pach	(EPFL, Switzerland and Rényi Institute, Hungary)
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Fun with Fonts: Algorithmic Typography

Erik D. Demaine*

Abstract

Over the past decade, my father and I have designed several typefaces based on mathematical theorems and open problems, specifically computational geometry [DD15, DD]. These typefaces expose the general public in a unique way to intriguing results and hard problems in hinged dissections, geometric tours, origami design, computer-aided glass design, physical simulation, protein folding, juggling, card shuffling, and alignment via folding. Most of our typefaces include puzzle fonts, where reading the intended message requires engaging in the mathematics itself, solving a series of puzzles which illustrate the challenge of the underlying mathematical problem.

Figures 1, 2, 3, and 4 show a few examples.

To play with the fonts, visit http://erikdemaine.org/fonts/

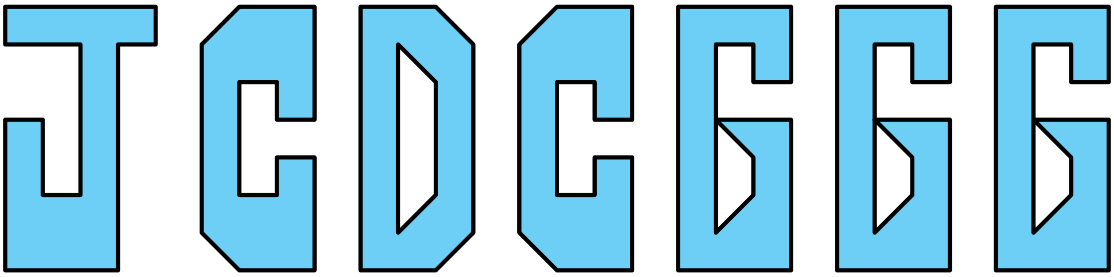

Figure 1: Hinged dissection font [DD03].

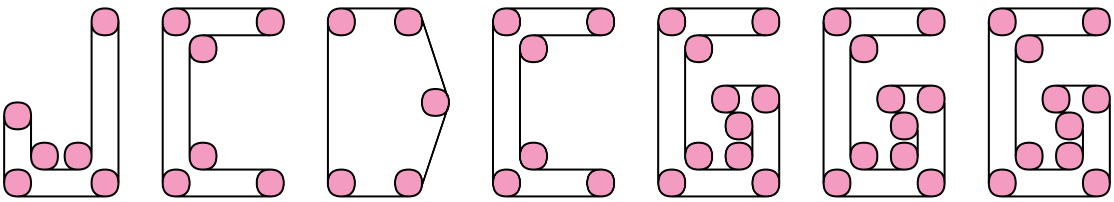

Figure 2: Conveyer belt font [DDP10a, DDP10b].

*MIT Computer Science and Artificial Intelligence Laboratory, 32 Vassar St., Cambridge, MA 02139, USA, edemaine@mit.edu

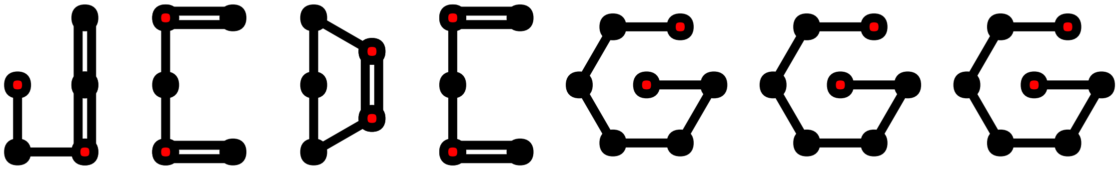

Figure 3: Linkage font [DD14].

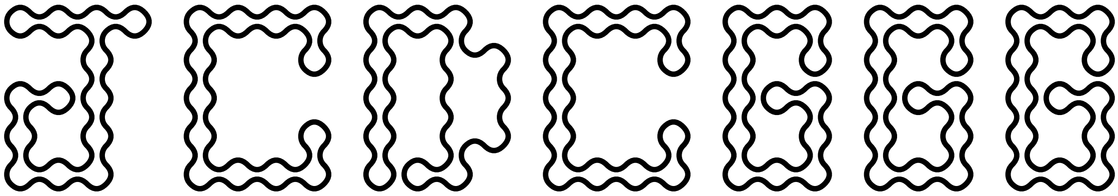

Figure 4: Tangle font [DDH+].

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Computational Design of Rigid Origami

Tomohiro Tachi*

1 Folding Motion

A sheet of paper is a material that cannot stretch, shrink or sheer, but can easily bend or fold. The behavior of such material is governed by the folding. If we closely observe the folding process of different patterns, we may notice some subtle differences. Some folding motions involve *crease-rolling*, which is a continuous change in the intrinsic position of the creases, and some folding motions are free of crease-rolling. Figure 1 compares folding motions of conventional bellows and a novel foldable tube. Notice that the conventional bellows fold with crease-rolling, while the other tube can fold without it.

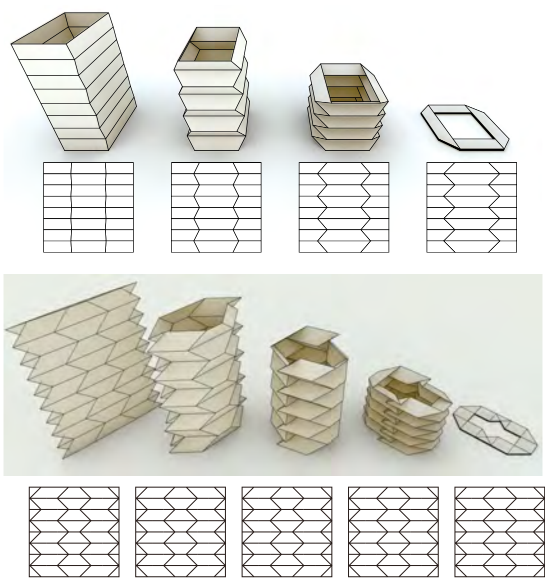

Figure 1: Comparison of folding motions. Top: The folding process of traditional bellows involves crease-rolling. Bottom: Rigid foldable tube folds without crease-rolling.

This difference becomes particularly critical when we consider engineering application of origami, where we are folding and unfolding repeatedly or using stiff materials such as metal sheets. In this case, crease-rolling will lead to the failure of the material. To make the folding mechanisms work robustly, it is crucial to find a continuous folding without crease-rolling. When we assume piecewise linear origami, a motion without crease-rolling is modeled as the mechanism formed by rigid panels and rotational hinges connecting them together. Such a model is called *rigid origami*, and its folding motion is called *rigid folding*. Rigid origami can lead to folding-based fabrication, deployable structures in architectural to micro scale, foldable antennas and solar panels for space, self-folding meta-materials, and self-reconfigurable robots.

2 Rigid Foldability

To design and analyze rigid origami, one of the most important problems is *rigid foldability*, a problem of judging if a given crease pattern can continuously transform from a flat state to another state by rigid folding. Because of piecewise isometry, rigid origami forms a system of multiple variables (fold angles) with multiple non-linear constraints (the closure along the chain around each vertex). Such a system is comparably easier to analyze in *generic* cases, but it is substantially harder to characterize when we consider *non-generic* cases. Here, a randomly generated polyhedral surface is expected to be generic with 100% probability. Nevertheless, most of the interesting behaviors of origami originate from non-generic cases.

A natural non-generic case is an origami in a flat unfolded state, which is a singular point of the solution space, where some geometric constraints become degenerate. There exists an asymmetry of folding and unfolding around such singular point. For example, an origami with the single interior vertex is known to be always continuously unfoldable from a folded state to the flat state [SW05], but only some of the flat origami patterns, characterized by [ACD$^+$16], can be folded to a 3D state. The flat state of origami can also be a bifurcation point: when driving force moment is applied at each crease, it may unpredictably choose one of the possible folding modes [TH16]. Each vertex has such choice of folding modes; for the rigid foldability of a general crease pattern, we need to consider whether there is a valid combination of these choices. In fact, we may reduce NP problems to rigid foldability problems, and thus rigid foldability is at least NP-hard [ACD$^+$].

3 Designing Rigid Origami Mechanisms

Another important special case of origami is called *overconstrained mechanism*, which is a system with a valid rigid folding motion but with more constraint equations than variables. An example of overconstrained mechanism is Miura-ori (Figure 2 Left), where its symmetry induces the degeneracy of con-

*The University of Tokyo, tachi@idea.c.u-tokyo.ac.jp

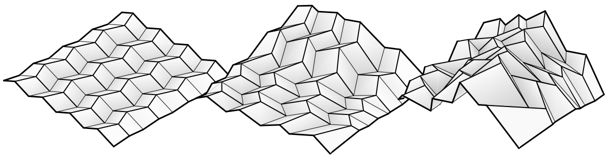

Figure 2: Variations of overconstrained Miura-ori

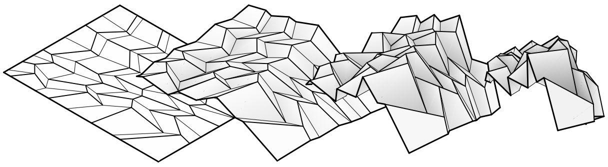

Figure 3: Folding motion of a variation in Figure 2.

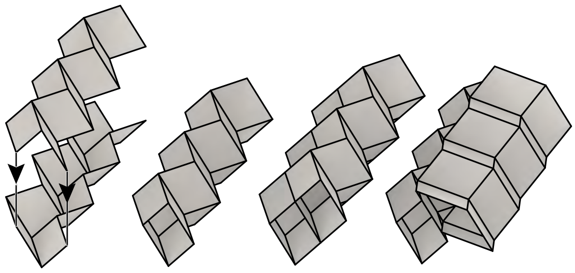

Figure 4: (a)Miura-ori [Miu70] combines into a (b) Rigid foldable tube [Tac09b]. Rigid foldable tubes can be coupled by translation [Tac09b] (c) or by (d) glide reflection [FTP15](zipper origami tube). Only (d) exhibits high stiffness.

straints to make it a mechanism. We may efficiently design variations of such rigid foldable origami patterns using a sufficient condition of rigid foldability: (1) every vertex is degree-4, (2) an opposite angle pair at each vertex sums up to π, and (3) it is not flat [Tac09a] . Figure 2 shows variations of Miura-ori that satisfies above conditions. They form overconstrained mechanisms (Figure 3).

We can further exploit non-generic origami patterns by combining mutually compatible rigid foldable mechanisms. For example, we may construct a rigid foldable origami homeomorphic to a cylinder by connecting a pair of rigid foldable sheets (Figure 1 Bottom or Figure 4(b)). Furthermore, we can combine multiple origami tubes into another rigid foldable structure (Figure 4). By the combination of multiple parts, the structure sometimes obtains high mechanical performance, i.e., high stiffness against its weight. When fabricated with paper, the structure with glide reflecting tubes (zipper tubes) (Figure 5) is flexible along the rigid folding motion and 400 times stiffer against undesirable deformation modes. This is a close-to-perfect foldable structure; the actuation at one end will be instantaneously transmitted to the other end.

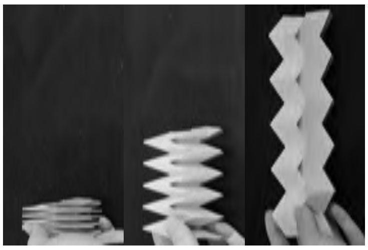

Figure 5: Zipper origami tube actuated from one end.

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Crossing the line

János Pach[*]

Given a set of (geometric) objects, their *intersection graph* is a graph whose vertices correspond to the objects, two vertices being connected by an edge if and only if their intersection is nonempty. Intersection graphs of intervals on a line [H57], more generally, chordal graphs and comparability graphs, turned out to be *perfect graphs*, that is, for them and for all of their induced subgraph H, we have $\chi(H) = \omega(H)$, where $\chi(H)$ and $\omega(H)$ denote the chromatic number and the clique number of H, respectively. It was shown [HS58] that the complements of these graphs are also perfect, and based on these results, Berge [B61] conjectured and Lovász [Lo72] proved that the complement of every perfect graph is perfect. By now, we have a complete characterization of all perfect graphs, which immediately implies the Lovász theorem.

Most geometrically defined intersection graphs are not perfect. However, in many cases they still have nice coloring properties. For example, Asplund and Grünbaum [AG60] proved that every intersection graph G of axis-parallel rectangles in the plane satisfies $\chi(G) = O((\omega(G))^2)$. The best known lower bound for $\chi(G)$ is linear in $\omega(G)$. For intersection graphs of chords of a circle, Gyárfás [G85] established the bound $\chi(G) = O((\omega(G))^2 4^{\omega(G)})$, which was improved to $O(2^{\omega(G)})$ in [KoK97]. Here we have a slightly superlinear lower bound. In some cases, there is no functional dependence between χ and ω. The first such example was found by Burling: there are sets of axis-parallel boxes in $\mathbb{R}^3$, whose intersection graphs are *triangle-free* ($\omega = 2$), but their chromatic numbers are arbitrarily large. Following Gyárfás and Lehel [GL83], we call a family $\mathcal{G}$ of graphs χ-*bounded* if there exists a function f such that all elements $G \in \mathcal{G}$ satisfy the inequality $\chi(G) \leq f(\omega(G))$. The function f is called a *bounding function* for $\mathcal{G}$. Heuristically, if a family of graphs is χ-bounded, then its members can be regarded "nearly perfect". Consult [G87, Ko04] for surveys.

At first glance, one might believe that, in analogy to perfect graphs, a family of intersection graphs is χ-bounded if and only if the family of their complements is. Burling's above mentioned constructions show that this is not the case: the family of complements of intersection graphs of axis-parallel boxes in $\mathbb{R}^d$ is χ-bounded with bounding function $f(x) = O(\log^{d-1} x)$. More recently, Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [PKK14] have proved that Burling's triangle-free graphs can be

[*]Ecole Politechnique Fédérale de Lausanne and Rényi Institute, Hungarian Academy of Sciences, P.O.B. 127 Budapest, 1364, Hungary; `pach@renyi.hu`. ; `pach@cims.nyu.edu`. Supported by Swiss National Science Foundation Grants 200021-165977 and 200020-162884.

realized as intersection graphs of segments in the plane. Consequently, the family of these intersection graphs is *not* χ-bounded either. On the other hand, the family of their complements is.

To simplify the exposition, we call the complement of the intersection graph of a set of objects their *disjointness graph.* That is, in the disjointness graph two vertices are connected by an edge if and only if the correspoinding objects are disjoint. Using this terminology, Larman, Matoušek, Pach, and Törőcsik proved the following result.

Theorem 1. [LMPT94] *The family of disjointness graphs of segments in the plane is χ-bounded.*

For the proof of Theorem 1, one has to introduce four partial orders on the family of segments four times. Although this method does not seem to generalize to higher dimensions, the statement does. We establish the following.

Theorem 2. P.-Tardos-Tóth [PTT16] *The family of disjointness graphs of segments in $\mathbb{R}^d, d \geq 2$ is χ-bounded.*

Theorem 3. P.-Tardos-Tóth [PTT16]

(i) For every n, there is a system of lines in $\mathbb{R}^3$ such that their disjointness graphs G_n satisfy $\lim_{n\to\infty} \frac{\chi(G_n)}{\omega(G_n)} = \infty$.

(ii) For infinitely many values of n, there is a system of n lines in $\mathbb{P}^3$ whose disjointness graph G'_n satisfies $\chi(G'_n) \geq 2\omega(G'_n) - 1$.

A continuous arc in the plane is called a *string.* One may wonder whether Theorem 1 can be extended to disjointness graphs of strings in place of segments. The answer is no, in a very strong sense.

Theorem 4. P.-Tardos-Tóth [PTT16] *There exist triangle-free disjointness graphs of n strings in the plane with arbitrarily large chromatic numbers. Moreover, we can assume that these strings are polygonal paths consisting of at most 4 segments.*

The following problems remain open.

Problem 5.

(i) Is the family of disjointness graphs of strings in the plane, any pair of which intersect in at most one point, χ-bounded?

(ii) Is this true for strings that are polygonal paths consisting of at most k segments, where $k > 1$ is fixed?

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Erdős-Szekeres type problems in integer grids

Ruy Fabila-Monroy

July 26, 2016

Abstract

In this talk we consider the problem of realizing, with integer coordinates of small size, well known point configurations in the plane. These sets are the extremal configurations of many well known Erdős-Szekeres type problems. We have proved lower bounds on the size of the integer grid needed to realize these configurations. Therefore, it is of interest to consider these problems in an integer grid such that their extremal configurations cannot be realized.

Let S be a set of n points in general position in the plane. A *convex k-gon* of S is a set of k points of S that are the vertices a convex polygon; a *k-hole* is a convex k-gon of S, such that no other point of S lies in the interior of this polygon. In 1935, Erdős and Szekeres proved the following theorem.

Theorem 0.1 (Erdős-Szekeres [6]). *For every positive integer k there exists a sufficiently large integer $n(k)$ such that the following holds. Every set of at least $n(k)$ points in general position in the plane contains a convex k-gon.*

Behind this result is the idea that every sufficiently large set of points contains a certain "ordered" configuration. The same underlying principle is behind Ramsey's theorem:

Theorem 0.2 (Ramsey [12]). *For every positive integer k there exists a sufficiently large integer $r(k)$ such that the following holds. Every complete graph of at least $r(k)$ vertices whose edges are colored with one of two colors, contains a complete monochromatic subgraph of k vertices.*

Given that $n(k)$ exists, it is of interest to find the largest point set in general position without a convex k-gon. Such a extremal configuration of 2^{k-2} points, was constructed by Erdős and Szekeres in [7].

The Erdős-Szekeres theorem has inspired many similar questions, in which one wonders whether every sufficiently large point set contains a certain special configuration. For example, in 1978 [5] Erdős asked if for every $k \geq 3$, any sufficiently large set of points in general position in the plane contains a k-hole. Shortly after, Harborth [9] showed that every set of 10 points contains a 5-hole. Horton [10] constructed arbitrarily large point sets without 7-holes, and thus without k-holes for larger values of k. His construction is now known as the Horton set. The case of 6-holes remained open for almost 30 years, until Nicolás [11], and independently Gerken [8], proved that every sufficiently large set of points contains a 6-hole.

For every such question there are extremal configurations—sets with a large number of points that do not contain the special configuration, or arbitrarily large point sets that do not contain it. In many instances essentially only one such configuration is known.

In recent papers [2, 1, 4, 3] we have studied the algorithmic problem of realizing these and other configurations with integer coordinates. Our main motivation behind this endeavor is to produce instances of these sets for other related problems. For computational reasons it is best if the coordinates of these sets are small as possible. We were able to prove good bounds on the size of the smallest integer grid in which these configurations can be realized. It is of interest to consider these Erdős-Szekeres type problems when restricted to integer grids small enough such that their known extremal configuration cannot be realized. In this talk we present these bounds and other results.

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Delone Sets: From Congruence of Local Patterns Towards Global Symmetry

Nikolay Dolbilin*
Steklov Mathematical Institute, Moscow
dolbilin@mi.ras.ru

In the talk we present some new results of the local theory of regular point systems. As known, the local theory aims to find conditions on a Delone set to be a regular point system.

We recall that a point set $X \subset \mathbb{R}^d$ is called a Delone (Delauney) set if for some positive numbers r and R two following conditions hold:
a ball $B_y(r)$ of radius r centered at any pt $y \in \mathbb{R}^d$ contains *at most one* pt $x \in X$;
a ball $B_y(R)$ of radius R centered at any pt y contains *at least one* pt $x \in X$.

Delone himself called such sets (r, R)-systems. Delone sets is a quite adequate model of the atomic structure of a condensed matter. However, such structures as crystals are described in terms of Delone sets of a special sort, namely, in terms of so-called regular and multi-regular systems. A Delone set X is called a *regular system* if its symmetry group $\mathrm{Sym}(X)$ is point-transitive, i.e. for any two points x and x' of X there is an isometry g such that $g(x) = x'$ and $g(X) = X$.

In the celebrated book "Geometry and the Imagination"of Hilbert and Cohn-Vossen a regular system is described as a Delone set X that *looks the same from any its point up to infinity.* 'Looks the same up to infinity' from any point, rigorously saying, means that the symmetry group acts on the set X transitively. Since the property 'looks the same up to infinity' obviously implies that the set X looks the same from any its point within the radius ρ for any $\rho > 0$. Saying more rigorously, the later one means that for any two points x and x' of the regular system X the isometry, which moves x to x' and the whole set X into itself, also moves a ρ-'neighborhood' $X \cap B_x(\rho)$ of the point x in the set X into a ρ-neighborhood $X \cap B_{x'}(\rho)$ of the point x' for any positive radius ρ. We will denote a ρ-neighborhood $X \cap B_x(\rho)$ by $C_x(\rho)$ and call it a ρ-*cluster* of the point x.

In this context the following question arises. Is there a value ρ_0 such that if in a Delone set X the ρ_0-clusters $C_x(\rho_0)$ at *all* points $x \in X$ are pairwise congruent then the set X is regular. In other words, whether an isometry g that moves the ρ_0-cluster $C_x(\rho_0)$ into the cluster $C_{x'}(\rho_0)$ moves also the entire set X into itself?

This question raises whenever you try to understand why under the phase transition from the liquid to the solid state the atomic structure of matter moves from an amorphous state into well-organized, periodic structure with a rich symmetry group. Physists explain (see e.g. Feynman R. et al (1964) Feynman Lectures on Physics, Vol. II, Ch.30) the transition from an amorphous state to a crystalline structure this

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way. Under crystallization process atoms of matter try to get stick together into local configurations on which the interaction energy between atoms of a given sort attains its minimum. So, it is natural to suppose that for a given sort of atoms there are one or few such optimal configurations. Thus, it is supposed that there are just few types of atomic fragments of some size to occur in an atomic structure over and over again.

However, it is not very clear why these local, physically explicable conditions imply periodicity of the global structure in three dimensions. The local theory for regular systems describes 'local rules' for a Delone set which imply its regularity (or multi-regularity). In other words, the local theory establishes regularity or multi-regularity of a Delone set from congruence of local fragments ('clusters'). Periodicity in d dimensions of a regular system, what is especially appreciated by physists, follows by the Shoenflies-Bieberbach theorem on existence of a translational subgroup with finite index in any crystallographic group.

In the talk we will discuss local criteria for regular [1] and multi-regular systems [2,5] and some their corollaries. Let $N(\rho)$ denote the *cluster counting function*, i.e. the number of congruent classes of ρ-clusters in a Delone set X. The cluster counting function is monotonically non-decreasing piece-wise constant function. In particular, $N(\rho) = 1$ means an important condition that all ρ-clusters $C_x(\rho)$ in a Delone set are pairwise congruent. It is clear that a Delone set X is a regular system if and only if $N(\rho) = 1$ for *all* $\rho > 0$.

In the talk we will select the following.
In plane ($d = 2$) a Delone set is a regular system if and only if $N(4R) = 1$ [3,5]
In space ($d = 3$) $N(10R) = 1$ *implies that a Delone set is a regular system.*
$N(4R) = 1$ *is non-refinable: for any* $\varepsilon > 0$ *there is a Delone set with* $N(4R - \varepsilon) = 1$ *which is not a regular system*, [3,5].

In the conclusion we will discuss recent results on *locally antipodal* Delone sets, i.e. Delone sets such that all $2R$-clusters are centrally symmetric [3,4,5]. Here we mention just two results on these sets.
1.*If for a locally antipodal Delone set* X $N(2R) = 1$ *then* X *is a regular system.*
2. *Any locally antipodal Delone set is the union of at most* $2^d - 1$ *congruent and parallel each other lattices.* Here d stands for dimension of space. This result implies that any locally antipodal Delone set is a multi-regular system.

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Signal Transmission, Ciphers and Polytopes

Ikuro Sato[1] and Jin Akiyama[2]

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1. Miyagi Cancer Center
   sato-ik510@miyagi-pho.jp
2. Tokyo University of Science
   ja@jin-akiyama.com

Extended Abstract

Regular and semi-regular polytopes are studied widely [1-19] but there are still many unknown parts remained. An n-dimensional polytope is called Wythoffian if it is derived by Wythoff construction from an n-dimensional regular polytope whose finite reflection groups belong to A_n, BC_n, F_4, G_2, H_3, H_4 and $I_2(p)$. Based on combinatorial and topological manner, we give a recurrence algorithm called Wythoff arithmetic to calculate the number of k-face ($k = 0 \sim n$) of all the Wythoffian n-polytopes using Schlaefli-Wythoff symbols. The correctness of the algorithm is reconfirmed by the methods of exhaustion using the computer.

In the n-digit of Wythoff 0/1-code, many important properties about n-Wythoffians like the numbers of k-faces, their shapes, volumes etc. are concealed. The bare bones of Wythoff code results from the mechanics of the Wythoff arithmetic, which is called the double flag architecture. To show the effectiveness of Wythoff arithmetic, let us take a 6-Wythoffian polytope P, which is expressed {33334} (010110) by Schlaefli-Wythoff symbol as an example (Fig. 1). Denoting by f_k the number of k-faces of P, it is calculated by the arithmetic;
$(a)\ f_0 = 5760,\ f_1 = 23040,\ f_2 = 32160,\ f_3 = 19680,\ f_4 = 5276$ and $f_5 = 476$ globally and $(b)\ f_0 = 1,\ f_1 = 8,\ f_2 = 22,\ f_3 = 29,\ f_4 = 20$ and $f_5 = 7$ around each vertex of P (See Appendix).

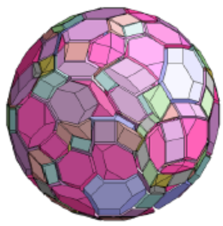

Fig. 1

By applying Wythoff arithmetic, 4 kinds of space-fillers (Wigner-Seitz cell in crystallography) in an arbitrary n-space are constructed:
[1] Minkowski tiles with $2(2^n - 1)$ facets
[2] BCC tiles with $2^n + 2n$ facets
[3] FCC tiles with $2n(n-1)$ facets
[4] HCP tiles with $n(n+1)$ facets

In this talk, not only the analysis of Wythoff arithmetic is shown but its possible applications for signal transmission, ciphers are presented.

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Appendix

$$\begin{bmatrix} 960 & 96 & 0 & 0 & 0 & 0 & 0 \\ 2880 & 192 & 0 & 0 & 0 & 0 & 0 \\ 2960 & 120 & 24 & 0 & 0 & 0 & 0 \\ 1200 & 24 & 36 & 4 & 0 & 0 & 0 \\ 162 & 1 & 14 & 4 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 12 \\ -60 \\ 160 \\ 240 \\ 192 \\ 64 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 4 & 0 & 0 & 0 & 0 & 0 \\ 13 & 6 & 1 & 0 & 0 & 0 & 0 \\ 13 & 4 & 3 & 1 & 0 & 0 & 0 \\ 6 & 1 & 3 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Geometric Algorithms for Lattice-based Modular Robots

Vera Sacristán

Departament de Matemàtiques
Universitat Politècnica de Catalunya & BGSMath
Barcelona, Spain
vera.sacristan@upc.edu

Introduction. Robots are becoming ubiquitous in our societies: industrial robots are widely used in manufacturing, storing and logistics, home robots vacuum floors and clean windows, medical robots perform surgery and introduce drugs in our bodies, humanoid robots train humans and accompany them, military robots fight in all sorts of ways, and also robots discover the most distant planets.

Within this continuously evolving context, modular robots are envisaged to become very useful multi-task robots [3]. These robots are made of small units (modules) that can attach and detach from each other, and move relative to each other. In this way, they can adopt different shapes, perform different tasks, and self-repair when a failure happens. These abilities make them a very promising tool, as they are expected to robustly adapt to different (previously unknown) environments and tasks, and to be reusable.

Modular robots are sometimes called self-reconfigurable [3] —hence emphasizing their capability of changing shape—, and also self-organizing [1] —to emphasize their distributed nature—. Directly related to their shape reconfiguration capability, geometry plays an important role in studying module's shapes and moves, and proposing convenient geometric solutions. Directly related to their self-organizing capability, efficient algorithms are crucial. An approach from computational geometry is, therefore, very appropriate.

In this talk I will present an overview of the geometric characteristics of the current modular robot prototypes in terms of their shapes and the geometric constraints for their movements, and I will revise the main reconfiguration and locomotion algorithms currently available.

The talk will be structured as follows.

Geometric abstraction of shapes and moves. There exists a traditional taxonomy of architectures, which classifies modular robots into chain, lattice and hybrid. I will start discussing this classification from the viewpoint of the underlying geometry, and then focus on shapes and movements of lattice-architecture modular robots. I will end this section discussing the reduction between the different movement paradigms by means of meta-modules.

Reconfiguration algorithms. I will very briefly present the current state of the art for reconfiguration algorithms for lattice-based modular robots (for a basic reference, see [2]), mainly focusing on the most important geometric issues related to them, and I will discuss what can/should be expected as new results for the variate modular shapes and moving paradigms.

Locomotion algorithms. Locomotion can be seen as a particular case of reconfiguration. Nevertheless, its specificity makes it possible and desirable to obtain locomotion algorithms that are simpler and more efficient than all-purpose reconfiguration algorithms. I will very briefly present the current state of the art for locomotion of lattice-based modular robots, focusing on their geometric aspects, and discuss locomotion-related open problems and future strategies.

Conclusions. I will end presenting open problems, possible algorithmic strategies, and geometric design suggestions for future modular robots.

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Even $1 \times n$ Edge-Matching and Jigsaw Puzzles are Really Hard

(Extended Abstract)

Jeffrey Bosboom[1], Erik D. Demaine[1], Martin L. Demaine[1], Adam Hesterberg[1], Pasin Manurangsi[2], and Anak Yodpinyanee[1]

[1]Massachusetts Institute of Technology, Cambridge, MA 02139, USA
{jbosboom,edemaine,mdemaine,achester,anak}@mit.edu
[2]University of California, Berkeley, CA 94720, USA
pasin@berkeley.edu

Jigsaw puzzles [9] and edge-matching puzzles [5] are two ancient types of puzzle, going back to the 1760s and 1890s, respectively. Jigsaw puzzles involve fitting together a given set of pieces (usually via translation and rotation) into a desired shape (usually a rectangle), often revealing a known image or pattern. The pieces are typically squares with a pocket cut out of or a tab attached to each side, except for boundary pieces which have one flat side and corner pieces which have two flat sides. Most jigsaw puzzles have unique tab/pocket pairs that fit together, but we consider the generalization to "ambiguous mates" where multiple tabs and pockets have the same shape and are thus compatible.

Edge-matching puzzles are similar to jigsaw puzzles: they too feature square tiles, but instead of pockets or tabs, each edge has a color or pattern. In *signed* edge-matching puzzles, the edge labels come in complementary pairs (e.g., the head and tail halves of a colored lizard), and adjacent tiles must have complementary edge labels on their shared edge (e.g., forming an entire lizard of one color). This puzzle type is essentially identical to jigsaw puzzles, where complementary pairs of edge labels act as identically shaped tab/pocket pairs. In *unsigned* edge-matching puzzles, edge labels are arbitrary, and the requirement is that adjacent tiles must have identical edge labels. In both cases, the goal is to place (via translation and rotation) the tiles into a target shape, typically a rectangle.

A recent popular (unsigned) edge-matching puzzle is *Eternity II* [8], which featured a US$2,000,000 prize for the first solver (before 2011). The puzzle remains unsolved (except presumably by its creator, Christopher Monckton). The best partial solution to date [7] either places 247 out of the 256 pieces without error, or places all 256 pieces while correctly matching 467 out of 480 edges.

Previous work. The first study of jigsaw and edge-matching puzzles from a computational complexity perspective proved NP-hardness [2]. Four years later, unsigned edge-matching puzzles were proved NP-hard even for a target shape of a $1 \times n$ rectangle [3]. There is a simple reduction from unsigned edge-matching puzzles to signed edge-matching/jigsaw puzzles [2], which expands the puzzle by a factor of two in each dimension, thereby establishing NP-hardness of $2 \times n$ jigsaw puzzles. Unsigned $2 \times n$ edge-matching puzzles were claimed to be APX-hard [1], but the proof is incorrect.[1]

Our results. We prove that $1 \times n$ jigsaw puzzles and $1 \times n$ edge-matching puzzles are both NP-hard, even to approximate within a factor of 0.9999999762 ($> \frac{41899199}{41899200}$). This is the first correct inapproximability result for either problem. Even NP-hardness is new for $1 \times n$ signed edge-matching/jigsaw puzzles. By a known reduction [2], these results imply NP-hardness for polyomino packing (exact packing of a given set of polyominoes into a given rectangle) when the polyominoes all have area $\Theta(\log n)$; the previous NP-hardness proof [2] needed polyominoes of area $\Theta(\log^2 n)$.

We prove inapproximability for two different optimization versions of the problems. First, we consider placing the maximum number of tiles without any violations of the matching constraints. This objective has a simple $\frac{1}{2}$-approximation for $1 \times n$ puzzles: alternate between placing a tile and leaving a blank. Second, we consider placing all of the tiles while maximizing the number of compatible edges between

[1]Personal communication with Antonios Antoniadis, October 2014. In particular, Lemma 3's proof is incomplete.

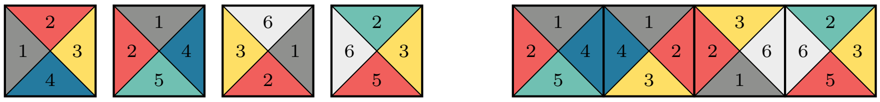

Figure 1: An example of a 1×4 (unsigned) edge-matching puzzle consisting of 4 tiles (left), where all tiles can be assembled into a 1×4 rectangular grid, with matching colors on the edges of adjacent tiles (right).

adjacent tiles (as in [1]).[2] This objective also has a simple $\frac{1}{2}$-approximation for $1 \times n$ puzzles, via a maximum-cardinality matching on the tiles in a graph where edges represent having any compatible edges: any solution with k compatibilities induces a matching of size at least $k/2$ [1]. Thus, up to constant factors, we resolve the approximability of these puzzles.

Our NP-hardness reduction is from Hamiltonian path on directed graphs whose vertices each has maximum in-degree and out-degree 2, which was shown to be NP-complete by Plesník [6]. To prove inapproximability, we reduce from a maximization version of this Hamiltonicity problem, called *maximum vertex-disjoint path cover*, where the goal is to choose as many edges as possible to form vertex-disjoint paths. (A Hamiltonian path would be an ideal path cover, forming a single path of length $|V| - 1$.) This problem is known to be NP-hard to approximate within some constant factor [4].

We prove that maximum vertex-disjoint path cover satisfies a stronger type of hardness, called *gap hardness*: it is NP-hard to distinguish between a directed graph having a Hamiltonian path versus one where all vertex-disjoint path covers having at most $0.999999284\,|V|$ ($> \frac{1396639}{1396640}\,|V|$) edges, given a promise that the graph falls into one of these two categories. This gap hardness immediately implies inapproximability within a factor of 0.999999284, although this constant is weaker than the known inapproximability [4]. More useful is that our reduction to $1 \times n$ jigsaw/edge-matching puzzles is gap-preserving, implying gap hardness and inapproximability for the latter. This approach lets us focus on "perfect" instances (where all tiles are compatible) versus "very bad" instances (where many tiles are incompatible), which seems far easier than standard L-reductions used in most inapproximability results, where we must distinguish between an arbitrary optimal and a factor below that arbitrary optimal.

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[2] A dual objective would be to place all tiles while minimizing the number of mismatched edges, but this problem is already NP-hard to distinguish between an answer of zero and positive, so it cannot be approximated.

Uniform Distribution on Pachinko*

Naoki Kitamura† Yuya Kawabata† Taisuke Izumi†

July 1, 2016

Pachinko is a mechanical gambling game popular in Japan. The player fires a ball into a vertically standing two-dimensional playfield, which is filled by many pins, spinners, and/or several electronic gadgets (like as pinball games). The ball fired from the top falls to the bottom with the effect of those gadgets. The player gets a reward when the ball goes into a winning pocket. This paper considers a mathematical aspect of Pachinko machines, which is initiated in the book written by Akiyama and Ruiz [1], and further argued in a recent paper by Akitaya et al. [2]. It presents several mathematical models of Pachinko and related problems. In this paper we consider the simplest 50-50 model, which is stated as follows: The playfield is a grid of equilateral triangles, where we can put a pin at each lattice point. A ball falls from the top center of the field. Hitting a pin, it falls the right or left side of the pin with equal probability. See Figure 1.

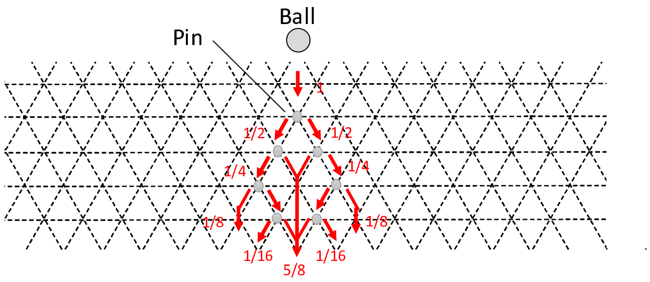

Figure 1: Mathematical Pachinko: 50-50 model

A layout of pins decides the probability that the ball falls on each column, which will naturally yields a inverse problem. That is, is there a pin layout whose output distribution follows a given probability distribution over n columns? The work by Akitaya et al. [2] solves this problem in part. They show that any distribution is approximately realizable with an arbitrary small error, and uniform distributions for $n = 2$, 4, 8, and 16 are exactly realizable. It is also conjectured that for any integer $k > 0$, uniform distributions of $1/2^k$ are realizable. Unfortunately, the known results for $k \leq 4$ are all found by exhaustive search by computers, and thus there is no algorithmic way of generating the pin layout of the uniform distribution for a given n. In this paper, we positively answer the open problem stated above. Precisely, we show that for any integer $k > 0$, the uniform distribution for $1/2^k$ is realizable. Our result is constructive, and the size of the pin layout is polynomial of n.

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†Nagoya Institute of Technology, Gokiso-cho, Showa-ku, Nagoya, Aichi, 466-8555, Japan. E-mail: t-izumi@nitech.ac.jp.

Search-and-Fetch on a Disk ⋆ ⋆⋆

Konstantinos Georgiou[1], George Karakostas[2], and Evangelos Kranakis[3]

[1] Ryerson University, Dept. of Mathematics, Toronto, Ontario, Canada.
[2] McMaster University, Dept. of Computing and Software, Hamilton, Ontario, Canada.
[3] Carleton University, School of Computer Science, Ottawa, Ontario, Canada.

Geometric search is concerned with finding a target placed in a geometric region and has been investigated in many areas of mathematics, theoretical computer science, and robotics. In each instance one aims to provide search algorithms that optimize a certain cost, which may take into account a variety of important characteristics and features of the domain, computational abilities of the searcher, assumptions about the target, etc.

Research Investigations We initiate the study of a new problem on *searching and fetching* in a distributed environment concerning *treasure-evacuation* from a unit disk. A treasure and an exit are located at unknown positions on the perimeter of a disk and at known arc distance. A team of two robots start from the center of the disk, and their goal is to fetch the treasure to the exit. At any time the robots can move anywhere they choose on the disk, independently of each other, with the same speed. A robot detects an interesting point (treasure or exit) only if it passes over the exact location of that point. We are interested in designing distributed algorithms that minimize the worst-case treasure-evacuation time, i.e. the time it takes for the treasure to be discovered and brought (fetched) to the exit by any of the robots.

The communication protocol between the robots is either *wireless*, where information is shared at any time, or *face-to-face* (i.e. non-wireless), where information can be shared only if the robots meet. For both models we obtain upper bounds for fetching the treasure to the exit. Our main technical contribution pertains to the face-to-face model. More specifically, we demonstrate how robots can exchange information without meeting, effectively achieving a highly efficient treasure-evacuation protocol which is minimally affected by the lack of distant communication. Finally, we complement our positive results above by providing a lower bound in the face-to-face model.

Search-and-Fetch is challenging even for a single robot. In this case we differentiate how the robot's knowledge of the distance between the two interesting points affects the overall evacuation time. We demonstrate the difference between knowing the exact value of that distance versus knowing only a lower

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⋆⋆ Evangelos Kranakis is the presenter.

bound and provide search algorithms for both cases. In the former case we give an algorithm which is off from the optimal algorithm (that does not know the locations of the treasure and the exit) by no more than $\frac{4\sqrt{2}+3\pi+2}{6\sqrt{2}+2\pi+2} \leq 1.019$ multiplicatively, or $\frac{\pi}{2} - \sqrt{2} \leq 0.157$ additively. In the latter case we provide an algorithm which is shown to be optimal.

Related Research Searching for a stationary point target has some similarities with the lost at sea problem, [8,9], the cow-path problem [2,3], and with the plane searching problem [1]. Our model is related to the recent works [4,5,6] investigating algorithms in the wireless and non-wireless (or face-to-face) communication models for the evacuation of a team of robots. Note that in this case, the "search domain" is the same and the evacuation problem without a treasure for a single robot is trivial. Thus, in addition to searching for the two stationary objects (namely treasure and exit) at unknown locations in the perimeter of a cycle we are also interested in fetching the treasure to the exit. As such, our search-and-fetch type problem is of much different nature than the series of evacuation-type problems above, and in fact solutions to our problem require a novel approach. More recent work can also be found in [7].

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Improved lower bounds on book crossing numbers of K_n

B. M. Ábrego*, J. Dandurand†‡, S. Fernández-Merchant*‡, E. Lagoda§, Y. Sapozhnikov†

California State University, Northridge

Abstract

A *k-page book drawing* of a graph G is a drawing of G on k halfplanes in the space with common boundary l, a line, where the vertices are on l and the edges cannot cross l. The k-page book crossing number of G, denoted by $\nu_k(G)$, is the minimum number of edge-crossings over all k-page book drawings of G. We improve the lower bounds on $\nu_k(G)$ for all $k \geq 15$ and determine $\nu_k(G)$ whenever $2 < n/k \leq 3$. Our proofs rely on bounding the number of edges in convex graphs with small local crossing numbers.

1 Introduction

In a *k-page book drawing* of a graph, the vertices are placed on a line l and each edge is completely contained in one of k fixed halfplanes in the space whose boundary is l. The *k-page book crossing number* of a graph G, denoted by $\nu_k(G)$, is the minimum number of edge-crossings over all k-page book drawings of G. Book crossing numbers have been studied in relation to their applications in VLSI designs. We are concerned with the k-page book crossing number of the complete graph K_n. In 1964, Blažek and Koman [2] described k-page book drawings of K_n with few crossings. They described their construction in detail only for $k = 2$, explicitly computed its crossing number for $k = 2$ and 3, and implicitly conjectured that generalizations of these constructions to larger values of k achieved $\nu_k(K_n)$. In 1994, Damiani et al. [3] described constructions using adjacency matrices, and in 1996, Shahrokhi et al. [5] provided a geometric description of similar k-page book drawings of K_n. In 2013, de Klerk et al. [4] gave another construction whose number of crossings is

$$Z_k(n) := r \cdot F\left(\left\lfloor \frac{n}{k} \right\rfloor + 1, n\right) + (k - r) \cdot F\left(\left\lfloor \frac{n}{k} \right\rfloor, n\right)$$

where $F(q, n) = q(q^2 - 3q + 2)(2n - 3 - q)/24$ and $r = (n \bmod k)$. Then

$$\nu_k(K_n) \leq Z_k(n) = \left(\frac{2}{k^2}\left(1 - \frac{1}{2k}\right)\right)\binom{n}{4} + O(n^3).$$

All the constructions in [3], [5], and [4] generalize the original Blažek-Koman construction but are slightly different. They are widely believed to be asymptotically correct giving rise to the following conjecture.

Conjecture 1. *For any positive integers k and n, $\nu_k(K_n) = Z_k(n)$.*

Ábrego et al. [1] proved this conjecture for $k = 2$. The only other previously known values of $\nu_k(K_n)$ are $\nu_k(K_n) = 0$ for $k > \lceil n/2 \rceil$ and a few sporadic values for $n \leq 15$ and $k \leq 5$ [4]. We prove the conjecture for any k and n such that $2 < n/k \leq 3$ (Theorem 5), and give improved lower bounds for $n/k > 3$ (Theorem 4). Shahrokhi et al. [5] proved the lower bound

$$\nu_k(n) \geq \frac{n(n-1)^3}{296k^2} - \frac{27kn}{37} = \frac{3}{37k^2}\binom{n}{4} + O(n^3),$$

which was later improved by de Klerk et al. [4] to

$$\nu_k(K_n) \geq \begin{cases} \frac{3}{119}\binom{n}{4} + O(n^3) \text{ if } k = 4, \\ \frac{2}{(3k-2)^2}\binom{n}{4} \text{if } k \text{ even, } n \geq \frac{k^2}{2} + 3k - 1, \\ \frac{2}{(3k+1)^2}\binom{n}{4} \text{ if } k \text{ odd, } n \geq k^2 + 2k - \frac{7}{2}. \end{cases} \quad (1)$$

Using semidefinite programming, they further improved the lower bound for several values of $k \leq 20$. We improve their bounds for $15 \leq n \leq 20$ as well as the asymptotic bound (1) for every k (Theorem 6).

2 Maximum number of edges

Our results heavily rely on a different problem on convex graphs. Let G_n be the rectilinear drawing of K_n whose vertices are the vertices of the regular n-gon. A *convex graph* can be defined as any subdrawing of G_n. To study crossings, it is convenient to disregard the sides of the polygon as edges. Let

*[bernardo.abrego, silvia.fernandez]@csun.edu.
†[julia.dandurand.7, yakov.sapozhnikov.473]@my.csun.edu.
‡Supported by the NSF grant DMS-1400653.
§evgeniya.lagoda@gmail.com

D_n be obtained form G_n by removing the sides of the polygon. Let $e_\ell(n)$ be the maximum number of edges over all convex subgraphs of D_n such that each edge is crossed at most ℓ times. Brass et al. studied the problem of maximizing the number of edges over convex graphs satisfying certain crossing conditions. Functions equivalent to $e_\ell(n)$ for general drawings of graphs in the plane were studied by Ackerman, and Pach et al. We determined the exact values of $e_\ell(n)$ for $\ell \leq 3$ and any n.

Theorem 2. *For any $n \geq 3$,*

$$\begin{array}{rcl} e_0(n) & = & n-3, \\ e_1(n) & = & \frac{3}{2}(n-3)+\delta_1(n), \\ e_2(n) & = & 2(n-3)+\delta_2(n), \\ e_3(n) & = & \frac{9}{4}(n-3)+\delta_3(n), \end{array}$$

where

$$\delta_1(n) = \begin{cases} 1/2 & \text{if } 2|n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_2(n) = \begin{cases} 1 & \text{if } 3|(n-2), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\delta_3(n) = \begin{cases} -1/4 & \text{if } n \equiv 0 \pmod 4, \\ 1/2 & \text{if } n \equiv 1 \pmod 4, \\ 5/4 & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

3 Crossings in k-page books

For any integers $k \geq 1$, $n \geq 3$, and $m \geq 0$, define $L_{k,n}(m) = \frac{m}{2}n(n-3) - k\sum_{\ell=0}^{m-1} e_\ell(n)$.

Theorem 3. *Let $n \geq 3$ and $k \geq 3$ be fixed integers. Then, for all integers $m \geq 0$, $\nu_k(K_n) \geq L_{k,n}(m)$. The value of $L_{k,n}(m)$ is maximized by the smallest m such that $e_m(n) \geq \frac{n(n-3)}{2k}$.*

We explicitly state the best bounds guaranteed by Theorem 3 and the values of $e_\ell(n)$ in Theorem 2.

Theorem 4. *For any $k \geq 3$ and $n \geq 2k$,*

$$\nu_k(K_n) \geq \begin{cases} \frac{1}{2}(n-3)(n-2k) \text{ if } 2k < n \leq 3k, \\ (n-3)(n-\frac{5}{2}k) - \delta_1(n)k \\ \qquad \text{if } 3k < n \leq 4k, \\ \frac{3}{2}(n-3)(n-3k) - (\delta_1+\delta_2)(n)k \\ \qquad \text{if } 4k < n \leq \begin{cases} \lceil 4.5k \rceil - 1 \text{ if } 4|n, \\ \lfloor 4.5k \rfloor \text{ otherwise,} \end{cases} \\ 2(n-3)(n-\frac{27}{8}k) - (\delta_1+\delta_2+\delta_3)(n)k \\ \qquad \text{otherwise.} \end{cases}$$

The first part of Theorem 4 settles Conjecture 1 when $2 < \frac{n}{k} \leq 3$.

Theorem 5. *If $2 < \frac{n}{k} \leq 3$, then $\nu_k(K_n) = \frac{1}{2}(n-3)(n-2k)$.*

The bound in Theorem 4 becomes weaker as n/k grows. We use a different approach to improve this bound when n is large with respect to k. Using Theorem 3, for fixed k and for all $n \geq n' \geq 4$,

$$\frac{\nu_k(K_n)}{\binom{n}{4}} \geq \frac{\nu_k(K_{n'})}{\binom{n'}{4}} \geq \max_{\substack{1 \leq m \leq 4 \\ n' \geq 2k}} \frac{L_{k,n'}(m)}{\binom{n'}{4}}.$$

We use $n' = \lfloor \frac{81}{16}k \rfloor$, which optimizes the previous lower bound when $k \equiv$ 3, 11, 15, 18, 22, 30, 37, 41, 48, 56, 60 (mod 64) and is close to optimal for all other values of k. The universal bound given in Theorem 6 is obtained when $k \equiv 29 \pmod{64}$ and it is the minimum of the maxima over all classes mod 64. This result improves the asymptotic bound (1) for every k. In fact, the ratio of the lower to the upper bound on $\lim_{n\to\infty} \frac{\nu_k(K_n)}{\binom{n}{4}}$ is improved from approximately $\frac{1}{9} \approx 0.1111$ to $\frac{2024}{81^2} \approx 0.3089$.

Theorem 6. *For any $k \geq 3$ and $n \geq \lfloor 81k/16 \rfloor$, $\nu_k(n) \geq \left(\left(\frac{8}{9}\right)^4 \frac{1}{k^2} + \left(\frac{2}{3}\right)^{15} \frac{118}{k^3} + \Theta\left(\frac{1}{k^4}\right) \right) \binom{n}{4}$.*

Finally, using $n' = \lfloor \frac{81}{16}k \rfloor$, we improved the bounds in [4] (Table 4.3) for $15 \leq k \leq 20$.

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The rectilinear local crossing number of $K_{n,m}$

B. M. Ábrego*, K. Dondzila†, S. Fernández-Merchant*‡, E. Lagoda§, A. Olavarrieta§, S. Sajjadi† Y. Sapozhnikov†

California State University, Northridge

Abstract

We bound $\overline{\text{lcr}}(K_{n,m})$, the rectilinear local crossing number of the complete bipartite graph $K_{n,m}$, for every n and m. We completely determine $\overline{\text{lcr}}(K_{n,m})$ whenever $\min(n,m) \leq 4$.

1 Introduction

We are concerned with rectilinear drawings of the complete graph $K_{n,m}$. That is, drawings with n red vertices and m blue vertices in the plane, where every edge joining two different color vertices is drawn as a straight line segment. We also assume that any two of these edges share at most one point.

In general, the local crossing number of a graph G was defined by Ringel as follows (see Guy et al. [2], Kainen [3], and Schaefer [5]). The *local crossing number* of a drawing D of a graph G, denoted $\text{lcr}(D)$, is the largest number of crossings on any edge of D. The *local crossing number* of G, denoted $\text{lcr}(G)$, is the minimum of $\text{lcr}(D)$ over all drawings D of G. This is also known as the *cross-index* (Thomassen [6]). The equivalent definition for rectilinear drawings is the *rectilinear local crossing number* of G, denoted $\overline{\text{lcr}}(G)$, as the minimum of $\text{lcr}(D)$ over all rectilinear drawings D of G. Recently, Ábrego and Fernández-Merchant [1] completely determined $\overline{\text{lcr}}(K_n)$ using a *Separation Lemma* (see Lemma 2 in [1]).

The *crossing number* of a graph G, denoted by $\text{cr}(G)$, is the smallest number of crossings among all drawings of G. When this minimum is restricted to rectilinear drawings, we obtained the *rectilinear crossing number* of G, denoted by $\overline{\text{cr}}(G)$. The value of $\overline{\text{cr}}(G)$ can be used to bound $\overline{\text{lcr}}(G)$ (as done in [2] for drawings of K_n on the torus). Namely, adding the number of crossings of every edge over all edges of a graph G counts precisely twice the number of crossings of G. In our problem, this means that It follows that

$$\overline{\text{lcr}}(K_{m,n}) \geq \frac{2\,\overline{\text{cr}}(K_{m,n})}{mn}.$$

The Zarankiewicz Conjecture (Paul Turán, 1944), states that $\overline{\text{cr}}(K_{m,n}) = \text{cr}(K_{m,n}) = Z(m,n) := \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$, but this has only been proved when $\min(m,n) \leq 6$, and for $m = 7$ and $n \leq 10$. The current best published lower bound on $\overline{\text{cr}}(K_{m,n})$ is $0.86\,Z(m,n)$ by de Klerk et al. [4] and recently, Norine and Zwols announced the lower bound $0.905\,Z(m,n)$, but this has not been published. This would yield

$$\overline{\text{lcr}}(K_{m,n}) \geq \frac{0.905}{8} mn + \Theta(mn) > 0.113125\, mn + \Theta(mn).$$

If the Zarankiewicz Conjecture were true, we would have

$$\overline{\text{lcr}}(K_{m,n}) \geq mn/8 + \Theta(mn).$$

The Zarankiewicz drawing of $K_{m,n}$ with $Z(n,m)$ crossings (see Figure 1) has local crossing number $\left(\left\lfloor \frac{m}{2} \right\rfloor - 1\right)\left(\left\lfloor \frac{m}{2} \right\rfloor - 1\right)$ showing that

$$\overline{\text{lcr}}(K_{m,n}) \leq mn/4 + \Theta(mn).$$

*[bernardo.abrego, silvia.fernandez]@csun.edu.
†[kory.dondzila.981, seyed.sajjadi.947, yakov.sapozhnikov.473]@my.csun.edu.
‡Supported by the NSF grant DMS-1400653.
§[evgeniya.lagoda, alain.olavarrieta]@gmail.com

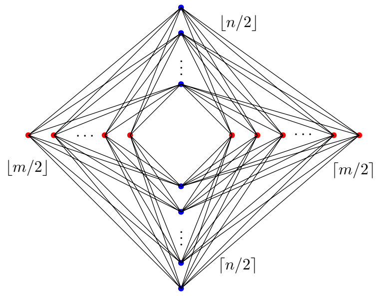

Figure 1: Zarankiewicz drawing of $K_{m,n}$ with $Z(m,n)$ crossings.

2 Main results

Clearly $\overline{\text{lcr}}(K_{2,n})$=0. We determine $\overline{\text{lcr}}(K_{n,m})$ when $\min(n,m) \leq 4$ and improve the upper bound for all other cases.

Theorem 1. *For any integer $n \geq 3$,*

$$\overline{\text{lcr}}(K_{3,n}) = \left\lceil \frac{n-2}{4} \right\rceil \text{ and } \overline{\text{lcr}}(K_{4,n}) = \left\lceil \frac{n-2}{2} \right\rceil.$$

Proof. (Sketch) Figure 2 shows a drawing of $K_{3,n}$ such that each edge is crossed at most $\lceil \frac{n-2}{4} \rceil$ times and there is an edge with that exact number of crossings. This shows that $\overline{\text{lcr}}(K_{3,n}) \leq \lceil \frac{n-2}{4} \rceil$. The red vertices form an equilateral triangle. There are two special blue points d and e very close to the top red point, one above and one below. The rest of the blue points are (almost) evenly distributed among four arcs of circle. The Zarankiewicz construction of $K_{4,n}$ for $m = 4$ and any n has local crossing number $\lceil \frac{n-2}{2} \rceil$ (see Figure 1) proving $\overline{\text{lcr}}(K_{4,n}) \leq \lceil \frac{n-2}{2} \rceil$. To prove that $\overline{\text{lcr}}(K_{3,n}) \geq \lceil \frac{n-2}{4} \rceil$, we consider several cases according to how the blue points are distributed among the regions determined by the red points. In each case, we identify 2 or 4 edges that must be crossed by a combined total of at least $\frac{n-2}{2}$ or $n-2$, respectively (see Figure 3). The proof of $\overline{\text{lcr}}(K_{4,n}) \geq \lceil \frac{n-2}{2} \rceil$ is more involved but follows similar lines. $\square$

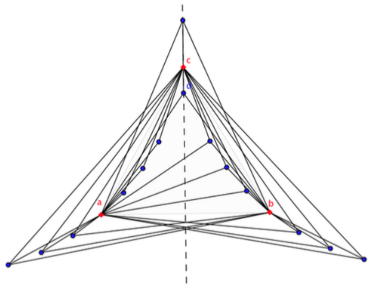

Figure 2: An optimal construction for $\overline{\text{lcr}}(K_{3,n})$.

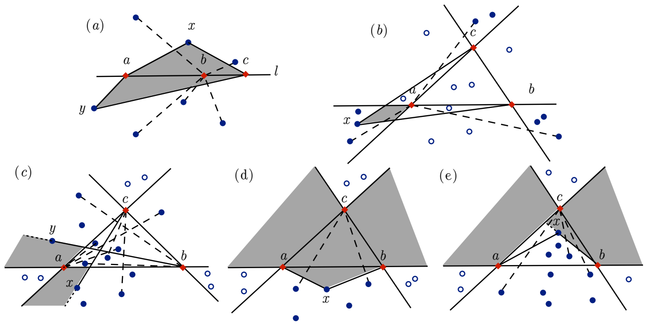

Figure 3: All shaded regions are empty of blue points.

Theorem 2. $\overline{\text{lcr}}(K_{m,n}) \leq \frac{2}{9}mn + \Theta(mn)$.

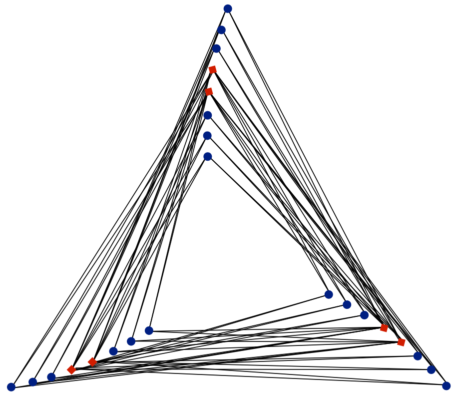

Figure 4: A drawing of $K_{m,n}$ with local crossing number $\frac{2}{9}mn + \Theta(mn)$. The red points are (almost) evenly distributed into 3 clusters and the blue into 6 clusters along 3 arcs of circle.

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New bounds on the maximum number of locally non-overlapping triangles in the plane.

Bernardo M. Ábrego* Silvia Fernández-Merchant* D. Flores-Peñaloza†

Abstract

We present new bounds on a problem by P. Brass on the maximum number of triangles that a drawing of a 3-uniform hypergraph in the plane can have so that any two triangles incident to a vertex do not have any other point in common.

1 Introduction

In his book, jointly co-authored with W. Mosser and J. Pach, P. Brass presented the following extremal geometric graph theory problem [3]. Related problems and results can be found in [1, 2].

Section 9.8, Problem 4 (Brass) What is the maximum number of hyperedges in a two-dimensional geometric three-hypergraph with n vertices in which no two edges incident to a vertex have any other point in common?

For concreteness, we denote that number as $f(n)$, and we denote the required property as the *local non-overlapping* property. It is observed that by choosing half of the interior faces of a maximal triangulation, one gets a set of triangles satisfying the local non-overlapping property. Thus, $f(n) \geq n-2$. By adding the interior angles of the triangles and observing that two triangles incident to a vertex must leave some uncovered angle between them, the upper bound $f(n) \leq 2n - O(1)$ is easily obtained.

In this note we narrow the gap between the two bounds by giving a better lower bound: $f(n) \geq 2n - c \cdot \log^3(n)$. We do this by first restricting the problem to the case when the set of n points is in convex position. We denote as $f^{conv}(n)$ the corresponding number. We give a construction that shows that $f^{conv}(n) \geq n - c \cdot \log_2^2(n)$. We then show how to adapt this convex-case construction to one for the general case, yielding our claimed bound.

We also list the exact values for $f^{conv}(n)$ for n up to 35.

2 Points in convex position

The problem of determining $f^{conv}(n)$, the maximum number of triangles that can be drawn with vertices on a set of n points in convex position, with the local non-overlapping property, turns out to be very interesting by itself. It is clear that any such set of triangles can be redrawn on the vertices of a regular n-gon, maintaining its local non-overlapping property. We thus might consider w.l.o.g. that the supporting point set is the set of vertices of a regular n-gon. The next easy lemma turns to be technically very helpful to tackle the problem. We denote the triangle with vertices $p, q,$ and r as $\Delta(p, q, r)$.

Lemma 1. $f^{conv}(n) \leq f^{conv}(n+1) \leq f^{conv}(n)+1$.

Proof. The left inequality is clearly true. We prove the second inequality.

Let T be a set of locally non-overlapping triangles whose vertices are those of a regular $(n+1)$-gon with circumcircle C. Let α be the maximum interior angle of any triangle in T. Let $t = \Delta(p, q, r)$ be one triangle s.t. $\angle rpq$ has size α. We claim that t is the only triangle incident to p. Suppose on the contrary that there exists another triangle $t' = \Delta(p, s, t)$ incident to p. Suppose w.l.o.g. that the points p, s, t, q, r appear in this order along the convex hull of the point set. Since the arc of C oposite to $\angle rpq$ is contained in the arc of C oposite to $\angle pst$, then $\angle pst$ is larger than $\angle rpq$, which has the maximum value α, a contradiction.

Now consider a set of $f^{conv}(n + 1)$ triangles on $n + 1$ points. By the previous observation, there exists one point p with only one triangle t incident to it. Remove both p and t to get a set of $f(n+1)-1$ triangles on n points. Thus, $f(n) \geq f(n+1)-1$. □

Theorem 1. $f^{conv}(n) \geq n - c \cdot \log^2(n)$.

[1]California State University, Northridge. [bernardo.abrego--silvia.fernandez]@csun.edu. S. Férnandez partially supported by the NSF grant DMS-1400653.

[2]Facultad de Ciencias, UNAM, México City. dflorespenaloza@ciencias.unam.mx. Partially supported by grants 168277 (CONACyT,Mexico), and PAPIIT IA102513 (UNAM, Mexico).

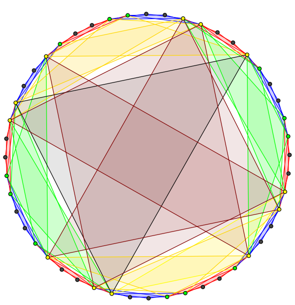
Proof (sketch). Let k be a positive integer and $n = 3(k+1)2^k$. We iteratively construct a set of n points with at least $n - (k+2)(k+1)$ locally non-overlapping triangles. This shows that $f(n) \geq n - (k+2)(k+1) > n - \log_2^2(n)$.

For stage t, we start with a set P_t of n_t points in convex position, where P_1 is the original set of points and $n_1 = n$. Write $n_t = 2t \cdot q_t + r_t$, where q_t and r_t are integers, and $0 \leq r_t < 2t$. Label the points in clockwise order $\{0, 1, 2, \ldots, n_t - 1\}$. For integers i and j, draw the triangle $T_{i,j} = \triangle(2t \cdot i + j, 2t \cdot i + j + t, 2t \cdot i + j + 2t)$, whenever $0 \leq i \leq q_t - 1$ and $0 \leq j \leq t - 1$. As long as n_t is even, let M_t be the set of points of the form $2t \cdot i + j + t$ where $0 \leq i \leq q_t - 1$ and $0 \leq j \leq t - 1$, or $2t \cdot q_t + j$ with $0 \leq j < r_t/2$. Let $P_{t+1} = P_t - M_t$. Then $n_{t+1} = |P_{t+1}| = n_t - tq_t - r_t/2 = 2t \cdot q_t + r_t - tq_t - r_t/2 = tq_t + r_t/2 = n_t/2$ and thus $n_{t+1} = n/2^t$. Also, stage t adds $tq_t = n_t/2 - r_t/2 = n/2^t - r_t/2 > n/2 - 2t$ triangles. Finally, relabel the points of P_{t+1} from 0 to $n_{t+1} - 1$, clockwise, in such a way that vertex 0 keeps its label throughout the process. Repeat this process until obtaining P_{k+1}, which has $n_{k+1} = n/2^k = 3(k+1)$ points labeled 0 to $3k+2$. Finally, draw the $k+1$ triangles $\triangle(i, i+(k+1), i+2(k+1))$ for $0 \leq i \leq k$. Since $\log_2 n = k + \log_2(3(k+1)) \geq k+2 > k+1$ for $k \geq 1$, then the total number of triangles in this construction is at least

$$\begin{aligned}
&\textstyle\sum_{t=1}^{k}(n/2^t - 2t) + (k+1) \\
&= \textstyle\sum_{t=1}^{k}(3(k+1)2^{k-t} - 2t) + (k+1) \\
&= n - (k+2)(k+1) > n - (\log_2 n)^2.
\end{aligned}$$

It can be verified that the constructed set of triangles has the local non-overlapping property. $\square$

The next figure shows the set of 88 triangles on 96 points given by the previous construction for $k = 3$. For clarity, every other point and its corresponding triangle formed with its neighboring points are omitted. Triangles and vertices are color-coded according to the stage they are added/removed. For complete detail, the figures can be arbitrarily zoomed-in in the digital version of this note.



To close the results for the convex case, we report the exact values for $f^{conv}(n)$ for n up to 35. These were obtained by mathematical analysis aided by computer search.

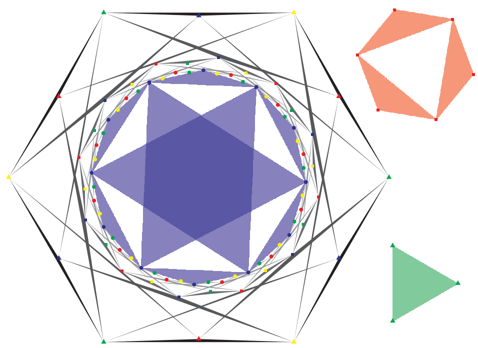
$$f^{conv}(n) = \begin{cases} n-2 & \text{for} \quad n = 3, \\ n-3 & \text{for} \quad 4 \leq n \leq 6, \\ n-4 & \text{for} \quad 7 \leq n \leq 12, \\ n-5 & \text{for} \quad 13 \leq n \leq 19, \\ n-6 & \text{for} \quad 20 \leq n \leq 35. \end{cases}$$

3 Points in general position

Theorem 2. *$f(n) \geq 2n - c' \log^3(n)$.*

Proof (sketch). Let k be a positive integer and $n = 3 \cdot 2^{k+2} - 12$. We construct a set of at least $n - 6 - 4\sum_{j=0}^{k-1} f^{conv}(3 \cdot 2^j) \geq 2n - c' \log^3(n)$ locally non-overlapping triangles on a set of n points. We carefully put together four copies of an optimal configuration for $f^{conv}(3 \cdot 2^j)$ for each $0 \leq j \leq k-1$, adding several thin triangles between copies. It can be verified that the constructed set of triangles has the local non-overlapping property.

The next figure shows part of this construction for $n = 84$ ($k = 3$). The complete construction consists of all the thin triangles shown in the figure plus 4 copies of each of the unique optimal configurations for $f^{conv}(3)$, $f^{conv}(6)$, and $f^{conv}(12)$ (each vertex pattern shown in the figure corresponds to one of these copies). $\square$



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The Sigma Chromatic Number of the Join of Graphs

Agnes D. Garciano
Maria Czarina T. Lagura
Reginaldo M. Marcelo[1]

Mathematics Department
Ateneo de Manila University, Quezon City, Philippines

Let G be a simple connected graph and $c : V(G) \to \mathbb{N}$ a vertex coloring of G in which adjacent vertices may have the same color. For any $v \in V(G)$, let $\sigma(v)$ be the sum of the colors of the vertices adjacent to v. Then c is called a *sigma coloring* of G if for any two adjacent vertices $u, v \in V(G)$, $\sigma(u) \neq \sigma(v)$. If $|c(V(G))| = k$ for a sigma coloring c, then G is said to be *sigma k-colorable*. The *sigma chromatic number* of G, denoted by $\sigma(G)$ is the least number of colors needed in a sigma coloring of G.

We define the *join* of two disjoint graphs G_1 and G_2 to be the graph $G = G_1 + G_2$ with

$$V(G) = V(G_1) \cup V(G_2) \quad \text{and} \quad E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}.$$

In this paper, we consider the join of graphs whose sigma chromatic numbers are known. Since any sigma coloring of the join $G_1 + G_2 + \cdots + G_m$ of disjoint graphs may be restricted to a sigma coloring of each of the graphs $G_1, G_2, \ldots, G_m$, it follows that $\sigma(G_1 + G_2 + \cdots + G_m) \geq \max\{\sigma(G_i) \mid 1 \leq i \leq m\}$. We investigate some conditions under which equality holds.

Suppose k_i and n_i are positive integers for $i = 1, \ldots, t$. Let $K_{k_1(n_1), k_2(n_2), \ldots, k_t(n_t)}$ denote the complete multipartite graph containing k_i partite sets of cardinality n_i for $i = 1, \ldots, t$. Then, $K_{k_1(n_1), k_2(n_2), \ldots, k_t(n_t)}$ is the join of $K_{k_1(n_1)}$, $K_{k_2(n_2)}$, $\ldots$, and $K_{k_t(n_t)}$. It was shown in [1] that

$$\sigma\left(K_{k_1(n_1), k_2(n_2), \cdots, k_t(n_t)}\right) = \max\left\{\sigma\left(K_{k_i(n_i)}\right) \mid 1 \leq i \leq t\right\}.$$

In this study, we will show that in general, if $G = G_1 + G_2 + \cdots + G_m$, for disjoint graphs $G_1, G_2, \ldots, G_m$, such that $\deg_G(u) \neq \deg_G(v)$ for any vertices $u \in G_i$ and $v \in G_j$ with $1 \leq i \neq j \leq m$, then

$$\sigma(G_1 + G_2 + \cdots + G_m) = \max\{\sigma(G_i) \mid 1 \leq i \leq m\}.$$

As an example, if G_i is either a path or a cycle of order $n_i > 6$, where $i = 1, 2, \ldots, m$ and $|n_i - n_j| \geq 2$ for $1 \leq i \neq j \leq m$, then

$$\sigma(G_1 + G_2 + \cdots + G_m) = \begin{cases} 3, & \text{if } G_i \text{ is an odd cycle for some } i, \\ 2, & \text{otherwise.} \end{cases}$$

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[1]Expected Presenter

Factors of bi-regular bipartite graphs

Yoshimi Egawa[1], Michitaka Furuya[2] and Mikio Kano[3]
[1] Tokyo University of Science,
[2] Kitasato University,
http://sites.google.com/site/michitakafuruya/
michitaka.furuya@gmail.com
[3] Ibaraki University
http://gorogoro.cis.ibaraki.ac.jp
mikio.kano.math@vc.ibaraki.ac.jp

For a graph G with vertex set $V(G)$, let $|G|$ denote the order of G, which is equal to $|V(G)|$. For a set $\mathcal{S}$ of integers, a spanning subgraph F of a graph G is called an *$\mathcal{S}$-factor* of G if $\deg_F(v) \in \mathcal{S}$ for every vertex v of G, where $\deg_F(v)$ denotes the degree of v in F. Thus for a set $\{a, b\}$ of two positive integers a and b, an $\{a, b\}$-factor F satisfies $\deg_F(v) = a$ or b for every vertex v. If $\mathcal{S}$ consists of one integer k, then an $\mathcal{S}$-factor is called a *k-regular factor* or briefly a *k-factor*.

It is known that if $a+3 \le b$, then the existence problem of an $\{a, b\}$-factor in a graph is NP-complete [4], and thus no nontrivial criterion for a graph to have such a factor is obtained. However, recently the following result on $\{a, b\}$-factors of regular graphs was obtained.

Theorem 1 (Akbari and Kano [1]) *Let r be an odd integer and a and b be positive integers such that $1 \le a \le b$ and $a + b = r$. Then the following hold.*

(i) If a is even and $2 \le a < r/2$, then every r-regular graph has an $\{a, b\}$-factor.

(ii) If a is odd and $r/3 \le a < r/2$, then every r-regular graph has an $\{a, b\}$-factor.

In this paper we consider similar factors in bipartite graphs, and show the following theorem.

Theorem 2 *Let s, t, c and r be positive integers such that $1 \le s \le t \le 2s+1$ and $(r/2)-1 \le c \le r/2$. Let G be a bipartite graph with bipartition (A, B) such that $\deg_G(x) = s+t$ for all $x \in A$ and $\deg_G(y) = r$ for all $y \in B$. Then G has a factor F that satisfies*

$$\begin{aligned} &\deg_F(x) \in \{s,t\} && \textit{for all } x \in A, \textit{ and} \\ &\deg_F(y) \in \{c, c+1\} && \textit{for all } y \in B. \end{aligned}$$

It is shown that the condition $(r/2)-1 \le c \le r/2$ in Theorem 2 is sharp when $s = t$.

We propose the following conjecture.

Conjecture 3 *Let s, t, c and d be positive integers such that $1 \le s \le t \le 2s$ and $1 \le c \le d \le 2c$. Let G be a bipartite graph with bipartition (A, B) such that $\deg_G(x) = s+t$ for all $x \in A$ and $\deg_G(y) = c+d$ for all $y \in B$. Then G has a factor F that satisfies*

$$\begin{aligned} &\deg_F(x) \in \{s,t\} && \textit{for all } x \in A, \textit{ and} \\ &\deg_F(y) \in \{c, d\} && \textit{for all } y \in B. \end{aligned}$$

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On the metric dimension of biregular graph

Suhadi Wido Saputro

Institut Teknologi Bandung, Indonesia

A set of vertices W *resolves* a graph G if every vertex in G is uniquely determined by its vector distance to the vertices in W. The *metric dimension* of G is the minimum cardinality of a resolving set of G. For integers $\mu, \sigma \geq 1$, a graph G is called a (μ, σ)-regular graph if every vertex of G is adjacent to μ or σ other vertices in G. In case of $\mu = \sigma$, we have a μ-regular graph (or σ-regular graph). In this paper, we determine the metric dimension of a connected (μ, σ)-regular graphs of order $n \geq 2$ where $1 \leq \mu \leq n-1$ and $\sigma = n-1$.

ON SAFE SETS OF THE CARTESIAN PRODUCT OF TWO COMPLETE GRAPHS

BUMTLE KANG, SUH-RYUNG KIM, AND BORAM PARK

1. Abstract

The notions of "safe set" and "connected safe set" have been introduced by Fujita *et al.* [2]. For a connected graph G, a set S of vertices in G is said to be a *safe set* if for every component C of the subgraph induced by S, $|C| \geq |D|$ holds for every component D of $G - S$ such that there exists an edge between C and D, and, especially, if the subgraph induced by S is connected, then S is called a *connected safe set*. For a connected graph G, the *safe number* $s(G)$ of G is defined as $s(G) = \min\{|S| \mid S \text{ is a safe set of } G\}$, and the *connected safe number* $cs(G)$ of G is defined as $cs(G) = \min\{|S| \mid S \text{ is a connected safe set of } G\}$.

The notions of safe set and connected safe set are motivated by the following problem. For a given topology of a building, it is required to place temporary accident refuges in addition to business spaces like discussion of conference rooms. Each temporary refuge should be available for the staff in every adjacent business space. (To mitigate the space cost, we assume that each temporary refuge will be used by the people in at most one of the adjacent business space.) Subject to the topology of the building being given, how can the temporary refuges be efficiently located so that the amount of business spaces is maximized?

Fujita *et al.* [2] showed that for a graph G

$$s(G) \leq cs(G) \leq 2s(G) - 1$$

and any tree T with at most one vertex of degree at least three satisfies the equality $s(T) = cs(T)$. For more recent work on this subject, the reader may refer to Bapat *et al.* [1]. Other than the trees mentioned above, the complete graphs obviously satisfy the equality. In this regard, we thought that it would be interesting to study which graphs satisfy the equality and Cartesian products of complete graphs are good to begin with as the Cartesian product operation preserves much of the structure of each of its factors. Here, the *Cartesian product* $G_1 \square G_2$ of two simple graphs G_1 and G_2 is a graph with vertex set $V(G_1) \times V(G_2)$ and having two vertices (u_1, u_2) and (v_1, v_2) adjacent if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2, or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1.

In this paper, by figuring out a way to reduce the number of components to two without changing the size of safe set, we shall show the following theorem:

Department of Mathematics Education, Seoul National University, Seoul 151-742, Korea
Department of Mathematics Education, Seoul National University, Seoul 151-742, Korea
Department of Mathematics, Ajou University, Suwon 16499, Korea
E-mail addresses: lokbt1@snu.ac.kr, srkim@snu.ac.kr, borampark@ajou.ac.kr.

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Theorem 1. *For two integers $n \geq m \geq 1$,*

$$s(K_m \Box K_n) = cs(K_m \Box K_n) = \alpha(m, n)$$

where $\alpha(1, n) = \lceil \frac{n}{2} \rceil$, $\alpha(2, n) = n$, and, for $m \geq 3$,

$$\alpha(m, n) = \min \left\{ mn - \sum_{i=1}^{2} m_i n_i + \max \left\{ \left\lceil \frac{\max\{m_1 n_1, m_2 n_2\} - mn + \sum_{i=1}^{2} m_i n_i}{2} \right\rceil, 1 \right\} \right\}$$

where the minimum is taken over the elements $((m_1, m_2), (n_1, n_2))$ in

$$P_2(m, n) = \{((m_1, m_2), (n_1, n_2)) \mid m_1 + m_2 = m, n_1 + n_2 = n, m_i, n_i \in \mathbb{N}\}.$$

Since $K_m \Box K_n$ is isomorphic to $K_n \Box K_m$, the condition $n \geq m \geq 1$ may be loosened into $m, n \geq 1$.

We present an algorithm for MATLAB computing $\alpha(m, n)$ in a polynomial time, which is written based on Theorem 1.

We prove Theorem 1 and we precisely formulate the safe number of $K_3 \Box K_n$ for $n \geq 3$ and $K_4 \Box K_n$ for $n \geq 4$.

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Title. On the competition graphs of d-partial orders

Authors. Jihoon Choi (presenter), Kyung Seok Kim, Suh-Ryung Kim, Jung Yeun Lee, and Yoshio Sano

Abstract. The *competition graph* of a digraph D is defined to be an undirected simple graph which has the same vertex set as D and which has an edge xy between two distinct vertices x and y if and only if for some vertex z, the arcs (x, z) and (y, z) are in D. Competition graphs was introduced by Cohen (1968) and have been extensively studied over last 40 years.

One of the main question in the field of competition graphs was to characterize the digraphs whose competition graphs are interval graphs. There was a lot of effort to answer this question, for example, Cohen (1978), Stief (1982), Lundgren and Maybee (1984), Hefner at al (1991). In 2005, Cho and Kim at last solved this problem. They defined digraphs so called "doubly partial order" and showed that the competition graph of a doubly partial order is an interval graph, and that every interval graph together with some isolated vertices is the competition graph of a doubly partial order.

In this talk, we extend the notion of doubly partial orders to the notion of d-partial orders. We study the competition graphs of d-partial orders and give their characterization. We also show that any graph can be made into the competition graph of a d-partial order for some positive integer d as long as adding isolated vertices is allowed. We then introduce the notion of the partial order competition dimension of a graph and study graphs whose partial order competition dimensions are at most three.

This is joint work with Kyung Seok Kim, Suh-Ryung Kim, Jung Yeun Lee, and Yoshio Sano.

Graph decomposition and labeling for RAID system

Tomoko Adachi
Department of Information Sciences, Toho University
2-2-1 Miyama, Funabashi, Chiba, 274-8510, Japan
E-mail: adachi@is.sci.toho-u.ac.jp

Keywords: Cluttered Ordering; Graph Decomposition; Label for a Graph

1. Introduction

The design of large disk array architectures leads to interesting combinatorial problems. The disk array architectures are known as *redundant arrays of independent disks* (RAID) (see [5]).

Some of these study replaced the problem of the RAID system in computer science, to the problem of cyclic orderings in graph theory. We pay the attention to constructions of cyclic orderings called *cluttered orderings*, which is introduced by Cohen et al. [4]. Mueller et al. [6] adapted the concept of wrapped Δ-labellings to the complete bipartite graph. Δ-labellings are a well-known tool for the decomposition of graphs into subgraphs (see [3]). As construction method, they defined the special bipartite graph $H(h;t)$, gave wrapped Δ-labellings of infinite family $H(1;t)$ and $H(2;t)$, and gave construction of cluttered ordering for the corresponding complete bipartite graph. Similarly, Adachi and Kikuchi [2] gave some label of infinite family $H(3;t)$, and Adachi [1] gave some label of infinite family $H(4;t)$. We consider this as the problem of decompositions of the complete bipartite graph and labelings of the special bipartite graph $H(h;t)$.

2. A Cluttered Ordering

Let $G = (V, E)$ be a graph with $n = |V|$ and $E = \{e_0, e_1, \cdots, e_{m-1}\}$. Let $d \leq m$ be a positive integer, called a *window* of G, and π a permutation on $\{0, 1, \cdots, m-1\}$, called an *edge ordering* of G. Then, given a graph G with edge ordering π and window d, we define $V_i^{\pi,d}$ to be the set of vertices which are connected by an edge of $\{e_{\pi(i)}, e_{\pi(i+1)}, \cdots, e_{\pi(i+d-1)}\}$, $0 \leq i \leq m-1$, where indices are considered modulo m. The cost of accessing a subgraph of d consecutive edges is measured by the number of its vertices. An upper bound of this cost is given by the *d-maximum access cost* of G defined as $\max_i |V_i^{\pi,d}|$. An ordering π is a (d, f)-*cluttered ordering*, if it has d-maximum access cost equal to f. We are interested in minimizing the parameter f.

In the following, $H = (U, E)$ always denotes a bipartite graph with vertex set U which is partitioned into two subsets denoted by V and W. Any edge of the edge set E contains exactly one point of V and W respectively. Let $\ell = |E|$, then a Δ-*labelling* of H with respect to V and W is defined to be a map $\Delta : U \to Z_\ell \times Z_2$ with $\Delta(V) \subset Z_\ell \times \{0\}$ and $\Delta(W) \subset Z_\ell \times \{1\}$, where each element of Z_ℓ occurs exactly once in the difference list

$$\Delta(E) := \Big(\pi_1(\Delta(v) - \Delta(w))\Big| v \in V, w \in W, (v, w) \in E\Big). \tag{2.1}$$

Here, $\pi_1 : Z_\ell \times Z_2 \to Z_\ell$ denotes the projection on the first component.

Definition 2.1 Let G be a graph with edge set $E(G) = \{e_0, e_1, \ldots e_{n-1}\}$, where n is positive integer, and let $\Sigma_0, \Sigma_1 \subset E(G)$ with $d := |\Sigma_0| = |\Sigma_1|$. For a permutation σ on $\{0, 1, \cdots, n-1\}$ define $V_i^{\sigma,d} := \bigcup_{j=0}^{d-1} e_{\sigma(i+j)}$ for $0 \le i \le n-d$. Then, for some given a positive integer f, and a map σ is called a (d, f)*-movement from* Σ_0 *to* Σ_1 if $\Sigma_0 = \{e_{\sigma(j)} | 0 \le j \le d-1\}$, $\Sigma_1 = \{e_{\sigma(j)} | n-d \le j \le n-1\}$, and $\max_i |V_i^{\sigma,d}| \le f$.

3. A bipartite graph $H(h;t)$

Let h and t be two positive integers. For each parameter h and t, we define a bipartite graph denoted by $H(h;t) = (U, E)$. Its vertex set U is partitioned into $U = V \cup W$ and consists of the following $2h(t+1)$ vertices:

$$\begin{aligned} V &:= \{v_i | 0 \le i < h(t+1)\}, \\ W &:= \{w_i | 0 \le i < h(t+1)\}. \end{aligned}$$

The edge set E is partitioned into subsets E_s, $0 \le s < t$, defined by

$$\begin{aligned} E'_s &:= \{\{v_i, w_j\} | s \cdot h \le i, j < s \cdot h + h\}, \\ E''_s &:= \{\{v_i, w_{h+j}\} | s \cdot h \le j \le i < s \cdot h + h\}, \\ E'''_s &:= \{\{v_{h+i}, w_j\} | s \cdot h \le i \le j < s \cdot h + h\}, \\ E_s &:= E'_s \cup E''_s \cup E'''_s, \quad \text{for } 0 \le s < t, \\ E &:= \bigcup_{s=0}^{t-1} E_s. \end{aligned}$$

We investigate labeling of $H(h;t)$ such that the corresponding complete bipartite graph has a cluttered ordering.

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Simple Folding is Really Hard

Hugo A. Akitaya* Erik D. Demaine† Jason S. Ku†

One of the most researched subsets of computational origami studies *flat foldings*—folded states of a polygonal paper that lie in a plane. If we unfold such a folding, we obtain a *flat-foldable crease pattern* which is the planar straight-line graph formed by the creases. Each crease originates from one of two types of fold: mountain (the paper folds backwards) or valley (the paper folds forwards). A crease pattern is called *assigned*, if each of its creases are labeled either mountain or valley, or *unassigned*, if no crease is labeled. The FLAT-FOLDABILITY problem asks whether a given crease pattern comes from some flat folding. This decision problem is known to be NP-complete for both assigned and unassigned crease patterns [2].

A *simple fold* is an operation that transforms a flat folding into another by a rigid 180° rotation of a subset of the paper around an axis ℓ. During the motion, the paper is not allowed to tear or self-cross. This restriction is motivated by practical sheet-metal bending, where a single robotic tool can fold the sheet material at once. The SIMPLE-FOLDABILITY problem asks whether a given crease pattern can be folded by a sequence of simple folds (unfolding is not allowed). Arkin et al. [1] introduced many models of simple folds with respect to the number of layers folded: they are *one-layer* (Fig. 1 (1), (5)), *all-layers* (Fig. 1 (1), (2), (3)), and *some-layers* (which imposes no restriction). They prove that SIMPLE-FOLDABILITY is weakly NP-complete for: one-layer (assigned), some-layers (assigned/unassigned) and all-layers (assigned/unassigned) if the paper is an orthogonal polygon and the creases are paper-aligned orthogonal (abbreviated ⊞); and for some-layers (assigned) and all-layers (assigned) if the paper is square and the creases are paper aligned at multiple of 45° (abbreviated ⊠). They also provide a polynomial-time algorithm for SIMPLE-FOLDABILITY with rectangular paper with paper-aligned orthogonal creases (abbreviated ⊞). Arkin et al. pose as an open problem whether there exist a pseudo-polynomial time algorithm for the models proven weakly NP-hard.

We settle this long standing open problem by proving strong NP-completeness for all models with ⊞ crease patterns (assigned/unassigned), and for some-layers and all-layers models with ⊠ crease patterns (assigned/unassigned). We reduce from 3-PARTITION, which is NP-complete [3]: can a set of integers $A = \{a_1, \ldots, a_n\}$ be partitioned into $n/3$ triples each with sum $\sum A/(n/3) = t$? Given an instance of 3-PARTITION, we construct the ⊞ crease pattern shown in Fig. 2, where $\infty = 10nt$. The vertical creases force the long vertical uncreased strip of paper on the left of the construction to pass through the right part. Collision is only avoided by folding through horizontal creases that encode the integers a_i if and only if the 3-PARTITION instance has a solution. To prove the results for ⊠ we create a crease pattern that forces the square paper to be folded into a long rectangular strip and then, using turn gadgets, into the orthogonal polygon shown in Fig. 2. We also point out an error in the NP-hardness proof in [1](Theorem 7.1) when the crease pattern is unassigned.

If it is hard to decide simple-foldability, a natural question arises: how close can we estimate the number of possible simple folds that can be performed? We define MAXFOLD, the natural optimization version of the decision problem asking for the maximum number of simple folds that can be folded given a crease pattern. We show that given a ⊞ crease pattern admitting a maximum sequence of m simple folds, approximating MAXFOLD to within a factor of $m^{1-\varepsilon}$ for any constant $\varepsilon > 0$ is NP-complete in the some-layers and all-layers models. To achieve such result, we transform the reduction in Figure 2 into a gap-producing reduction by adding $O(n^{1/\varepsilon})$ horizontal creases splitting the existing vertical creases. The new creases can only be

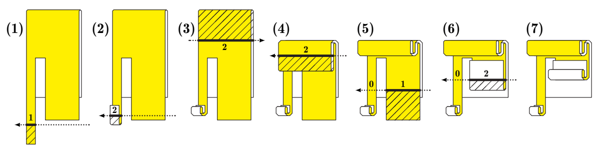

Figure 1: Example folding steps demonstrating the differences between simple folding models. The axis ℓ is a directed dotted line and the simple fold rotates the textured subset of the paper.

*Tufts University, hugo.alves_akitaya@tufts.edu
†MIT, {edemaine,jasonku}@mit.edu

folded if all the preexisting creases of the construction are already folded.

Additionally, we propose three new simple folds models, namely *infinite-one-layer*, *infinite-some-layers* and *infinite all-layers* that require that exactly one, at least one, or all layers are folded at the intersection of ℓ and the flat folding, respectively. Examples are shown in Fig. 1 (1), Fig. 1 (1), (3), (4), and Fig. 1 (1), (3), respectively. We prove strong NP-hardness for the infinite-one-layer and infinite-some-layers models by a reduction from 3-Partition. Given an instance of 3-Partition, we construct the ⊞ crease pattern shown in Fig. 3, where $\delta = \frac{3}{2n}$. The uncreased portion of the construction forces the creases labeled $c_i, i \in \{1, \ldots, 2\frac{n}{3}\}$ to be folded only when they are aligned with a crease labeled s_i after h_1 and h_2 are folded, or else the axis ℓ would intersect an uncreased region of the paper. Alignment is only possible by folding correct creases $v_j, j \in \{1, \ldots, n\}$, whose positions encode integers in A, if and only if the 3-Partition instance has a solution.

Finally, we show a polynomial-time algorithm for ⊞ crease patterns in all infinite models based on the algorithm in [1]. These results motivate why rectangular maps have orthogonal, not diagonal creases.

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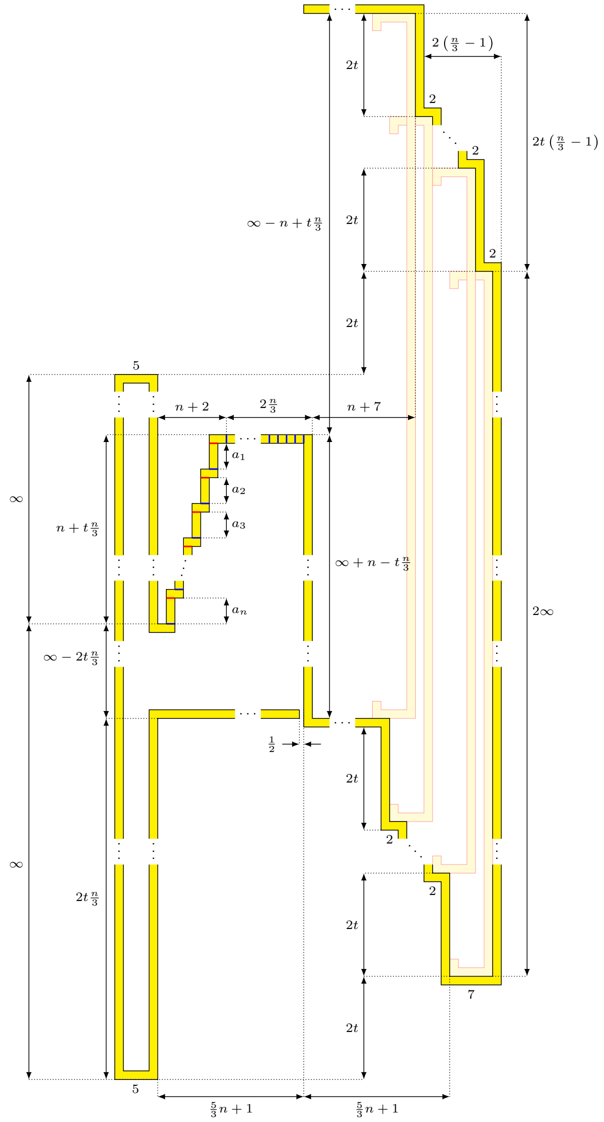

Figure 2: Reduction from 3-Partition to Simple-Foldability for one-layer, some-layers, and all-layers. Mountains and valleys are drawn in red and blue respectively.

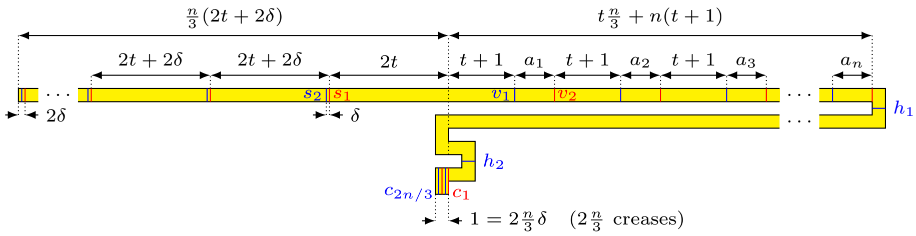

Figure 3: Reduction from 3-Partition to Simple-Foldability in the infinite-one-layer and infinite-some-layers models. Crease assignment is drawn in red and blue for mountain and valley respectively.

Satisfying Multiple Boundary Conditions

Erik D. Demaine* Jason S. Ku*

It has been shown that an isometry always exists to fold a paper to match a non-expansive folding of its boundary [1]. However, there is little (if any) research in designing crease patterns that satisfy multiple constraints. In this paper, we analyze crease patterns that can fold to multiple prescribed folded boundaries and flat-foldable states, such that every crease in the crease pattern is finitely folded in each folding.

Theorem 1 *Given a four cornered paper, there exists a single vertex crease pattern folding through each corner of the paper that also folds flat.*

Proof. A single degree-four vertex in a flat-foldable crease pattern must obey Kawasaki's theorem, that the sum of opposite angles sum to π. From this condition, one can derive a condition on possible positions (x, y) of the single vertex. We can parameterize any simple quadrilateral with cyclically ordered points $a = (-1, 0)$, $b = (x_1, y_1)$, $c = (1, 0)$, and $d = (x_2, y_2)$, where y_1 is positive and y_2 is negative, and the line from a to c is a visible diagonal. The condition on the location of a flat-foldable vertex is then given by the following cubic equation:

$$x(y_1 + y_2)(x^2 + y^2 - 1) - y(x_1 + x_2)(x^2 + y^2 + 1) + (x_1y_2 + x_2y_1)(y^2 - x^2 + 1) + 2xy(1 + x_1x_2 - y_1y_2) = 0. \tag{1}$$

The curve defined by this equation passes through each corner of the paper, as can be readily verified. However, we must prove that the curve passes through the interior of the paper. It suffices to show that the tangent to the curve at one of the vertices passes between its two adjacent edges. Taking partial derivatives of Equation 1, one can show the tangent to the curve at p_a has the same direction the following vector:

$$v_T = (\,(x_1 + 1)(x_2 + 1) - y_1y_2,\; y_1(x_2 + 1) + y_2(x_1 + 1)\,). \tag{2}$$

The edges adjacent to p_a have directions $v_b = (x_1+1, y_2)$ and $v_d = (x_2 + 1, y_2)$ respectively. Taking magnitude of the cross products in the $\hat{z}$ direction out of the plane yields the following relations:

$$(v_T \times v_b) \cdot \hat{z} = -(\,(1 + x_1)^2 + y_1^2\,)y_2; \tag{3}$$

$$(v_T \times v_d) \cdot \hat{z} = -(\,(1 + x_2)^2 + y_2^2\,)y_1. \tag{4}$$

Because y_2 is always negative, the first condition is always positive, so the top edge is a left turn from the tangent line. Because y_1 is always positive, the second condition is always negative, so the bottom edge is a right turn from the tangent line, so local to p_a, the curve must intersect the quadrilateral, completing the proof. □

For quadrilateral paper, the solution space of single vertex crease patterns satisfying a folding of its boundary is an ellipse on the interior of the paper. The equation of this ellipse in general is quite complicated. However, in the case of kite quadrilaterals, the ellipse is axis aligned with the diagonals, and for squares the ellipse is centered. Let the corners of the square be $(-1, 0)$, $(0, 1)$, $(1, 0)$ and $(0, -1)$. We parameterize the folding of the square boundary by the distances between the two diagonals, distance $2\sqrt{1 - a^2}$ along the x axis and distance $2\sqrt{1 - b^2}$ along the y axis; this parameterization will simplify the equations later on. Using this parameterization, the equation of the ellipse of crease pattern vertices satisfying the boundary condition (a, b) is as follows:

$$\frac{x^2}{a^2(1 - b^2)} + \frac{y^2}{b^2(1 - a^2)} = a^2 + b^2. \tag{5}$$

Since the ellipse is centered at the origin, it must cross the x and y axes four times except in the degenerate cases where the ellipse becomes a line or a point, specifically when a or b equal 1 or 0.

Theorem 2 *Given a square of paper and two foldings (a_1, b_1), (a_2, b_2) of its boundary folded only at the corners, then if the intervals $[a_1, b_1]$ and $[a_2, b_2]$ overlap, then there exists a single vertex crease pattern that folds exactly to both boundaries.*

Proof. The proof is by construction. The approach will be to calculate the set of possible crease patterns with one interior vertex that folds to each boundary, and show that the two sets have crease patterns in common when the intervals $[a_1, b_1]$ and $[a_2, b_2]$ overlap.

We have already shown that the solution space for each boundary condition is an ellipse given by Equation 5. Given two such ellipses parameterized by (a_1, b_1) and (a_2, b_2), they can be made to intersect as long as the smaller major radius is larger

*MIT, {edemaine,jasonku}@mit.edu

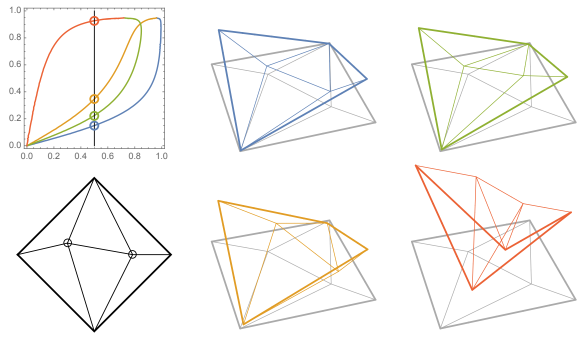

Figure 1: The state space of a two vertex crease pattern, plotting a vs. b, and four folded states for a single value of a.

than the minor radius of the other since the roles of a and b are interchangeable under boundary mappings. A simple yet tedious case analysis shows that this equation holds when intervals $[a_1, b_1]$ and $[a_2, b_2]$ overlap. The converse statement is not true as there are points when the intervals do not overlap such that the inequality is still true. □

Theorem 3 *Given any two nonexpansive boundary foldings of a square paper folding at its vertices, there exists a one or two-vertex crease pattern that can fold rigidly to meet both boundary conditions.*

Proof. The proof is by construction. The approach will be to calculate a subset of possible crease patterns with two interior vertices that folds to each boundary, and show that the two sets have crease patterns in common.

We will parameterize a subset of two-vertex crease patterns in the special case where one vertex resides on a diagonal. We will let s be the distance between this vertex p and point $(-1, 0)$. Solving the distance equations again yields the equation of an ellipse, this time of the following form:

$$\frac{(x - x_0)^2}{r_x^2} + \frac{y^2}{r_y^2} - 1 = 0. \tag{6}$$

However, this time there are two possible ellipses for each choice of boundary condition: one when the crease from $(-1, 0)$ to p is a valley fold, and one when the crease is a mountain fold. The parameters of the ellipse in each case are given by:

$$x_0 = \frac{sb^2(1 + b^2)}{2(a^2 + b^2)((1 - s) + b^2) \pm 2sb\sqrt{1 - a^2}\sqrt{a^2 + b^2}}; \tag{7}$$

$$r_x = b\sqrt{\frac{1 - a^2}{a^2 + b^2}} - \frac{sb^2(1 - b^2)}{2(a^2 + b^2)((1 - s) + b^2) \pm 2sb\sqrt{1 - a^2}\sqrt{a^2 + b^2}}; \tag{8}$$

$$r_y = a\sqrt{\frac{1 - b^2}{a^2 + b^2}}\left(1 - \frac{sb^2}{(a^2(1 - s) + b^2) \pm b\sqrt{1 - a^2}\sqrt{a^2 + b^2}}\right). \tag{9}$$

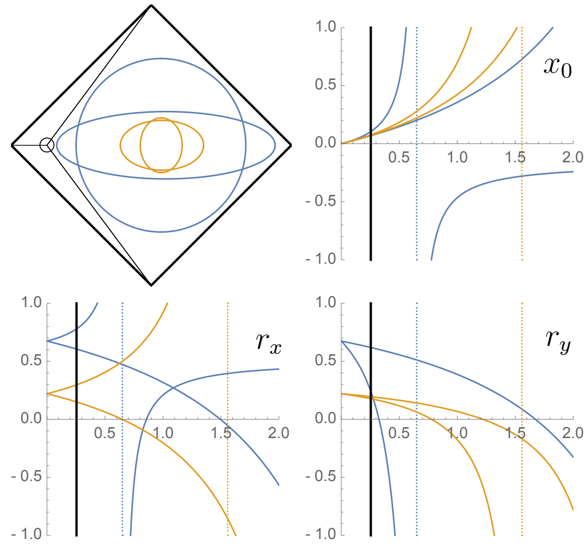

Figure 2: Graphs of how x_0, r_x, and r_y vary with respect to s for $(a_1, b_1) = (0.3, 0.3)$ (blue) and $(a_2, b_2) = (0.95, 0.95)$ (orange).

When $s = 0$, these parameters reduce Equation 6 to Equation 5. Figure 1 shows a plot of this state space for one such crease pattern. The bottom right corner corresponds to the flat state.

The equations continue to define an ellipse as long as r_y does not become negative. If the ellipse corresponding to (a_1, b_1) and (a_2, b_2) do not intersect at $s = 0$, then that means both r_x and r_y are larger for one and not the other because x_0 is zero. Without loss of generality, assume $a_1 > a_2$ and $b_1 > b_2$. r_y is zero precisely when:

$$s(r_y = 0) = 1 \pm b\sqrt{\frac{1 - a^2}{a^2 + b^2}}. \tag{10}$$

Since r_y is symmetric about 1 for any (a, b), this means r_y for (a_1, b_1) and r_y for (a_2, b_2) must be equal form some s. If they are equal, their corresponding ellipses must intersect, which corresponds to a two vertex solution. Figure 2 shows how x_0, r_x, and r_y vary with respect to s. □

We conjecture that any finite set of corner foldings of a square can be satisfied with a finite crease pattern such that every crease folds by a nonzero amount when satisfying each boundary folding.

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Folding and Punching Paper

Yasuhiko Asao* Erik D. Demaine† Martin L. Demaine† Hideaki Hosaka‡
Akitoshi Kawamura§ Tomohiro Tachi¶ Kazune Takahashi*

Abstract

We show how to fold a piece of paper and punch one hole so as to produce any desired pattern of holes.

1 Introduction

In the *fold-and-cut problem* introduced at JCDCG'98 [DDL98], we are given a planar straight-line graph drawn on a piece of paper, and the goal is to fold the paper flat so that exactly the vertices and edges of the graph (and no other points of paper) map to a common line. Thus, one cut along that straight line (and unfolding the paper) produces exactly the given pattern of cuts. This problem always has a solution [DO07, BDEH01], though so far the number of folds depends on both the number n of vertices and the ratio r of the largest and smallest distances between nonincident vertices and edges. (A rough estimate on the number of folds is $O(nr)$.)

In the *fold-and-punch problem*, we are given n points drawn on a piece of paper, and the goal is to fold the paper flat so that exactly those points (and no other points of paper) map to a common point. Thus, punching one hole at that point (and unfolding the paper) produces exactly the given pattern of holes. This problem is a natural analog of the fold-and-cut problem where we replace one-dimensional features and target (segments onto a common line) with zero-dimensional features and target (points onto a common point); thus, we also call the problem *zero-dimensional fold and cut*. This problem is also a special case of the *multidimensional fold-and-cut problem* posed in [DO07, after Open Problem 26.32].

Directly applying a fold-and-cut solution to the graph with n vertices and zero edges does not solve the corresponding fold-and-punch problem, because the n points would come to a common line but not a common point. This discrepancy can be fixed by then making $n-1$ one-layer simple folds along perpendicular bisectors between consecutive points (all perpendicular to the common line).

Our goal in this paper is to find more efficient algorithms for the fold-and-punch problem. Indeed, in all four variations described below, we find solutions that depend polynomially in n and only logarithmically or not at all on r (the ratio of the largest and smallest distances between points); see Table 1.

Problem 1 (0-dimensional fold and cut) *Given n points $p_1, p_2, \ldots, p_n$ on a piece of paper, find a flat folding f such that*

$$f(p_1) = f(p_2) = \cdots = f(p_n) \neq f(q) \text{ for all } q \neq p_i.$$

If such a folding exists, what is the order of the number of folds?

We have four variations of this problem based on the following two criteria:

1. Finite paper or infinite paper
2. Allow or forbid crease lines through given points

The second criterion is motivated by the observation that creases passing through given points may lead to a difficulty in the actual punching operation because it has zero tolerance; a small misalignment leads to missing hole or duplicated holes. For example, the fold-and-cut solution places creases passing through the given points.

Theorem 2 *Problem 1 is always solvable in all cases above. The orders of the number of folds (number of folding steps, each of which is composed of either a simple fold or a folding with $O(1)$ creases) and the number of resulting creases in the crease pattern are stated in the following table.*

	Crease Passing		Crease Not Passing	
	Folds	Creases	Folds	Creases
Finite	$O(n)$	$O(n)$	$O(n \log r)$	$O(n^2 r)$
Infinite	$O(n)$	$O(n^2)$	$O(n \log r)$	$O(n^2 r)$

Table 1: Results: Number of folds and resulting creases required in each of the four problem variants.

In the rest of this abstract, we show the sketch of proof of the following two cases: (1) finite paper, allowing crease passing and (2) infinite paper, forbidding crease passing.

2 Finite Paper, Allowing Crease Passing

The proof is by construction. The basic strategy is to align multiple points onto a single horizontal line by folding along horizontal creases and then to add bisectors between consecutive points as follows:

*University of Tokyo, {asao,kazune}@ms.u-tokyo.ac.jp
†MIT, {edemaine,mdemaine}@mit.edu
‡Azabu Junior-High School, h_hosaka314@hotmail.com
§University of Tokyo, kawamura@graco.c.u-tokyo.ac.jp
¶University of Tokyo, tachi@idea.c.u-tokyo.ac.jp

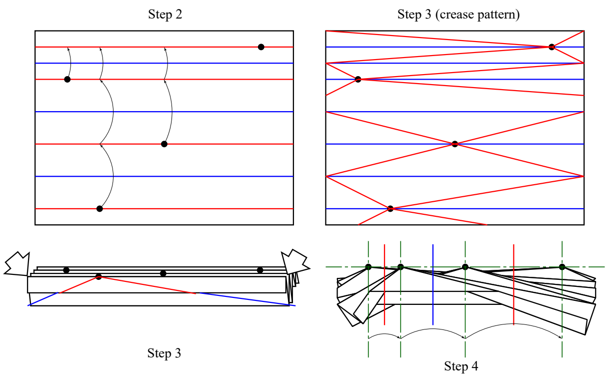

Figure 1: Steps 2–4 to fold given points to a point.

Step 1: Rotate By rotating the paper in the xy-plane, we may assume that the y coordinates $y_1, y_2, \ldots, y_n$ of $p_1, p_2, \ldots, p_n$, respectively, are distinct each other.

Step 2: Horizontally Align Assume $y_1 < y_2 < \cdots y_n$. Then fold the paper along

- mountain creases: lines $y = y_1, y = y_2, \ldots, y = y_n$, and
- valley creases: lines $y = (y_1 + y_2)/2, y = (y_2 + y_3)/2, \ldots, y = (y_{n-1} + y_n)/2$.

As a result, $p_1, p_2, \ldots, p_n$ are on mountain creases and aligned colinearly.

Step 3: Clear Overlaps There exists only one p_i's on each mountain crease. By folding along two slanted lines through p_i, no point except $p_1, \ldots, p_n$ is on the line which $p_1, \ldots, p_n$ are aligned.

Step 4: Vertically Fold Fold along the perpendicular bisectors of adjacent p_i's. This folds p_i's to a single point.

3 Infinite Paper, Forbidding Crease Passing

We introduce *upshifting gadget* to align p_i to a horizontal line while avoiding any part of the paper folded onto p_i. Figure 2 shows an upshifting gadget, which is composed of a pair of twist folds with width d and angle $\theta < 45°$ separating the paper into 6 regions except for the gaps of $3d$ between them. By folding this gadget, these regions get closer to each other. Also, the regions stay singly covered, avoiding other parts of the paper to overlap. If we fix upper-center region to a plane, upper-left(right) region moves to the right (left) by $2d$, bottom-left(right) region moves to upper-right(left) by $2\sqrt{2}d$, and the bottom-center region moves up vertically by $2d$. Here is the detailed steps that replace Steps 2 and 3 of finite crease-passing version.

Step A: Initialize Sort points by its height such that p_1 is the highest point. We draw a horizontal line ℓ passing through p_1. Now consider p_i, the highest point bellow ℓ. i is initially 2.

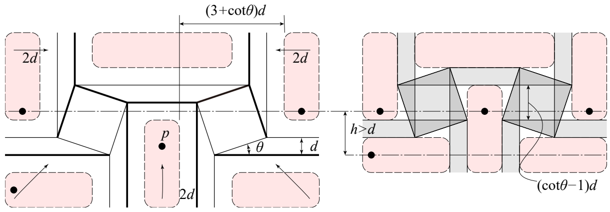

Figure 2: An upshifing gadget that shifts 6 regions painted pink.

Step B: Shrink Let h be the vertical separation between p_i and p_{i+1}. Let 2 the minimum horizontal separation from p_i to other point p_j ($j \neq i$). Add a horizontal pleat between ℓ and p_i until their distance $2d$ is strictly smaller than $\min(0.5w, 0.5h)$. Here, the number of folds required is at most $O(\log r)$.

Step C: Upshift Insert an upshifting gadget such that $p_0, \ldots, p_{i-1}$ are on either upper-left or upper-right region, p_i is in the bottom-center region, and $p_{i+1} \ldots$ are on either bottom-left or bottom-right region. Fold the gadget to align p_i to ℓ. Increment i and go to Step B until every point is on ℓ.

Combining with the same Steps 1 and 4, we can successfully fold p_i exclusively to a single point.

acknowledgment

We would like to thank Masaki Watanabe for the idea for solving finite paper crease passing case.

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Unfolding and Dissection of Multiple Cubes

Zach Abel[1] Brad Ballinger[2] Erik D. Demaine[1] Martin L. Demaine[1]
Jeff Erickson[3] Adam Hesterberg[1] Hiro Ito[4] Irina Kostitsyna[5]
Jayson Lynch[1] Ryuhei Uehara[6]

A *polyomino* is a "simply connected" set of unit squares introduced by Solomon W. Golomb in 1954. Since then, a set of polyominoes has been playing an important role in puzzle society (see, e.g., [3, 1]). In Figure 82 in [1], it is shown that a set of 12 pentominoes exactly covers a cube that is the square root of 10 units on the side. In 1962, Golomn also proposed an interesting notion called "rep-tiles": a polygon is a rep-tile of order k if it can be divided into k replicas congruent to one another and similar to the original (see [2, Chap 19]).

These notions lead us to the following natural question: is there any polyomino that can be folded to a cube and divided into k polyominoes such that each of them can be folded to a (smaller) cube for some k? That is, a polyomino is a *rep-cube* of order k if itself is a net of a cube, and it can be divided into k polyominoes such that each of them can be folded to a cube. If each of these k polyominoes has the same size, we call the original polyomino *regular* rep-cube of order k. In this paper, we give an affirmative answer. We first give some regular rep-cubes of order k for some specific k. Based on this idea, we give a constructive proof for a series of regular rep-cubes of order $36gk'^2$ for any positive integer k' and an integer g in $\{2, 4, 5, 8, 10, 50\}$. That is, there are infinitely many k that allow regular rep-cube of order k. We also give some non-regular rep-cubes and its variants.

We first show some specific solutions.

Theorem 1 *There exists a regular rep-cube of order k for $k = 2, 4, 5, 8, 9, 36, 50, 64$*

Proof. For each of $k = 2, 4, 5, 8, 9$, we give a regular rep-cube in Figure 1. It is not difficult to see that they satisfy the condition of rep-cubes.

For $k = 36$, we use six copies of the pattern given in Figure 2. Using this pattern, we can combine them into any one of eleven nets of a cube.

For $k = 64$, we use one copy of the left pattern in Figure 2 for the bottom of a big cube, four copies of the center pattern in Figure 2, and one copy of the right pattern in Figure 2 for the top of the big cube. The consistency can be easily observed.

For $k = 50$, we make a program for finding packings of nets of unit cubes on twisted grids on bigger cubes by exhaustive search. We found a packing on a $(7, 1)$ twist, i.e., a dissection of the surface of a $\sqrt{50}$ $\sqrt{50}$ $\sqrt{50}$ cube into 50 nets of unit cubes as shown in Figure 2. ∎

Based on the solution for $k = 36$ in Theorem 1, we obtain the following theorem:

Theorem 2 *There exists a regular rep-cube of order $36gk'^2$ for any positive integer k' and an integer g in $\{2, 4, 5, 8, 10, 50\}$. That is, there exists an infinite number of regular rep-cubes.*

Proof.(Outline) In each pattern in the proof of Theorem 1, we first split each unit square into k'^2 small squares. Then we replace each of them by the pattern for $k = 36$ in Figure 2, and obtain the theorem. ∎

One may think that non-regular rep-cubes are more difficult than regular ones. So far, we have found some:

Theorem 3 *There exists a non-regular rep-cube of order k for $k = 2, 10$.*

Proof. For $k = 2$, the rep-cube is given in Figure 3(left): this itself folds to a cube of size $\sqrt{5}$ $\sqrt{5}$ $\sqrt{5}$, and it can be cut into two pieces such that one folds into a cube of size 2 2 2, and the other folds into a unit cube. We note that these areas satisfy $6 \quad (\sqrt{5})^2 = 6 \quad 1^2 + 6 \quad 2^2 = 30$.

For $k = 10$, the rep-cube is given in Figure 3(right): this pattern contains 150 unit squares. It is easy to see that nice nets of unit cube use 54 unit squares in total. The remaining 96 squares form a net of cube of size 4 4 4. Moreover, this pattern also folds to a cube of size 5 5 5. These areas satisfy $150 = 6 \quad 5^2 = 6 \quad (3^2) + 6 \quad (4^2) = 6(3^2 + 4^2)$. ∎

In this paper, we introduce a new notion of "rep-cube," and show several examples. So far, we have no systematic ways to investigate them. However, from the trivial constraint for the areas, we can consider many variants as shown in the last example for $k = 10$: Is there a rep-cube of order 6 from a 3 3 3 cube into

[1] MIT, {zabel,edemaine,mdemaine,achester, jaysonl}@mit.edu
[2] Humboldt State Univ., ballingerb@gmail.com
[3] UIUC, jeffe@cs.uiuc.edu
[4] UEC, itohiro@uec.ac.jp
[5] ULB, irina.kostitsyna@ulb.ac.be
[6] JAIST, uehara@jaist.ac.jp

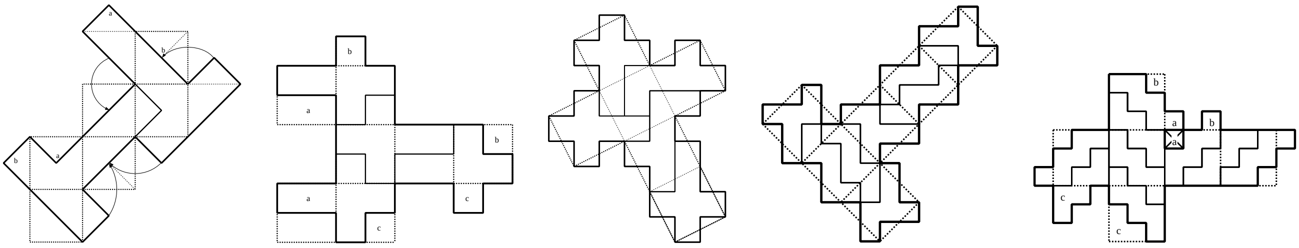

Figure 1: Rep-cubes of order $k = 2, 4, 5, 8, 9$, respectively.

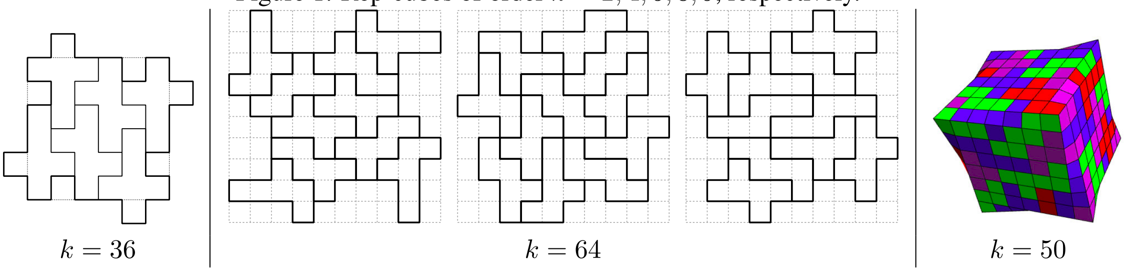

Figure 2: Patterns for rep-cubes of order $k = 36$, $k = 64$, and $k = 50$

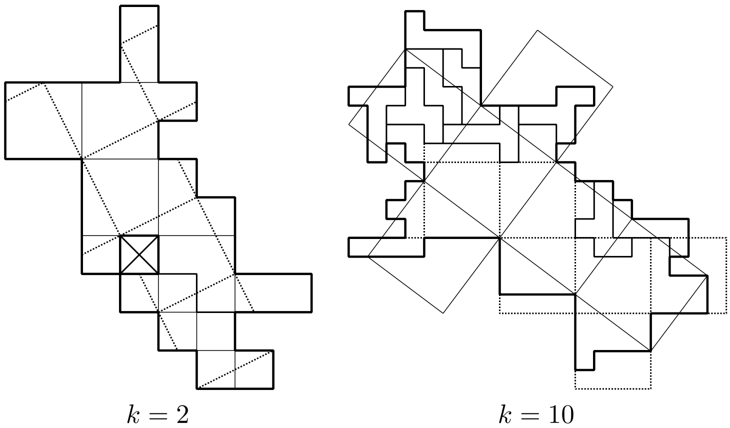

Figure 3: Patterns for non-regular rep-cubes of order $k = 2, 10$.

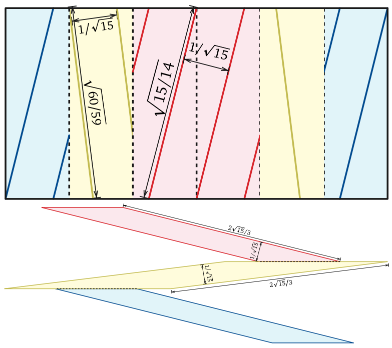

Figure 4: One doubly covered square to three doubly covered squares.

one 2   2   2 cube and five 1   1   1 cubes, and so on. Especially, one interesting open question is that is there are rep-cube of order 2 from one 5   5   5 cube into one 4   4   4 cube and 3   3   3 cube. We note that this size comes from the Pythagoras triangle $3^2 + 4^2 = 5^2$. We have already known that there are infinitely many Pythagoras triangles. For each of them, can we construct a rep-cube of order 2?

Is there any integer k such that we have no regular rep-cube of order k? It seems to be unlikely that there is a regular rep-cube of order 3. How can we prove that? In this paper, we also introduce "regular" rep-cubes. One natural additional condition may be making every small development congruent; for example, each example for $k = 2, 4, 9$ satisfies this condition. What happens if we employ this additional condition?

One of other extensions is different dimension and shape. For example, we have the following theorem:

Theorem 4 *Let $A, a_1, \ldots, a_k$ be any positive real numbers such that $\sum_i a_i = A$. (1) There is a net of a doubly-covered square with area A that can be cut into k polygons with areas $a_1, \ldots, a_k$, each of which can be folded into a double-covered square (see Figure 4 for $k = 3$). (2) There is a net of a regular tetrahedron with area A that can be cut into k polygons with areas $a_1, \ldots, a_k$, each of which can be folded into a regular tetrahedron.*

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Spanning trees with different diameters*

Eduardo Rivera-Campo
Departamento de Matemáticas
Universidad Autónoma Metropolitana-Iztapalapa.
erc@xanum.uam.mx

Abstract

Harary, Mokken and Plantholt [1] proved that if a and b are, respectively, the minimum and the maximum diameter of a spanning tree of a 2-connected graph G, then G contains a spanning tree with diameter c for each integer c with $a \leq c \leq b$. In this paper we prove that if $r \geq 2$ and G is a connected graph with minimum degree $r+2$, then G has a spanning tree with diameter $D-r+1$, where D is the length of a longest path in G.

Let G be a 2-connected graph with diameter D and let $P : x_0, x_1, \ldots, x_{s-1}, x_s, y_s, y_{s-1}, \ldots, y_1, y_0$ be a path of G with length D, where $s = \lfloor D/2 \rfloor$ and $x_s = y_s$ if D is odd. Consider the partition $N_s, N_{s-1}, \ldots, N_0$ of $V(G)$ obtained as follows:

$$N_s = \{x_s, y_s\}$$
and
$$N_{i-1} = \{x_{i-1}, y_{i-1}\} \cup \{z \in V(G) \setminus (V(P) \cup N_s \cup N_{s-1} \cup \cdots \cup N_i) : uz \in E(G) \text{ with } u \in N_i\}$$
for $i = s, s-1, \ldots, 1$.

By construction G has a spanning tree T with diameter D, containing path P and such that if $uv \in E(T)$, then either $uv = x_s y_s$ or u and v lie in consecutive levels N_i and N_{i-1}. We show that T can be transformed into a spanning tree T' of G with diameter $D-r+1$.

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On the Capture Time of Cops and Robbers Game on a Planar Graph

Photchchara Pisantechakool and Xuehou Tan
School of Science and Technology and School of Information Science
Tokai University, Hiratsuka, 259-2192, Japan

***Abstract*—This paper examines the capture time of a planar graph in a pursuit-evasion games' variant called the cops and robbers game. Since any planar graph is 3-cop-win, we study the capture time of a planar graph G of n vertices using three cops, which is denoted by $capt_3(G)$. We present a new capture strategy and show that $capt_3(G) \leq 2n$. This is the first result on $capt_3(G)$.**

I. Introduction

A pursuit-evasion game called the Cops and Robbers game is played on a finite connected undirected graph G by two players: a cop player and a robber player. The game has a set of rules. First the cop player and then the robber player occupy some vertex of G. After that they move alternatively along the edges of G. The cop player wins if she succeeds in capturing, i.e., occupying the same vertex as, the robber. The robber player wins if he can avoid being captured indefinitely. It is obvious that for every graph G, one of the players must win. In a graph where the cop player has a winning strategy against the robber, it is considered as cop-win. In this version of pursuit-evasion game, both players know the location of each other's piece. In a setting where multiple cops are allowed, a cop player controls them and may move them simultaneously in her turn.

The length of the game, or the capture time of a graph by j cops, denoted as $capt_j(G)$, has been studied recently. In 2009, Bonato et al.[2] considered the capture time of various cop-win graphs, and concluded that while the capture time of a cop-win graph of n vertices is bounded above by $n-3$, half the number of vertices is sufficient for a large class of graphs including chordal graphs. For the graphs with multiple cops required to win, the capture time can be calculated by a polynomial-time algorithm if the number of cops is fixed. In 2011, Mehrabian [3] showed that the capture time of grids (2-cop-win) is half the diameter of the graph, or $capt_2(G) \leq \lfloor \frac{m+n}{2} \rfloor - 1$ for $m \times n$ grid.

Aigner and Fromme's proved that for a planar graph G, three cops suffice to win the game (3-cop-win)[1]. They also provided necessary capturing method and concepts to prove their theorem. One concept is the assignment of a stage i in the capture strategy to a certain subgraph $R_i \subset G$, which contains all the vertices that the robber can safely enter. The graph R_i is called the *robber territory*. Another is a concept called the *guarded path*, which is the shortest path between two vertices such that a cop c can capture the robber if the robber r ever enter any vertex on that path. After one cop successfully controlled a path, the robber territory changes such that $R_{i+1} \subsetneq R_i$. Their method is to repeatedly find a new guarded path that differs from previous ones, until the robber territory is eventually reduced to one vertex. Since the guarded paths used in their strategy are not all distinct, the capture time can roughly be bounded by $n(n-1)$.

In this paper, we focus on the capture time of the cops and robbers game on a planar graph (3-cop-win) in general, which has not yet been studied well. We present a new capture strategy by refining the work of Aigner and Fromme[1] in the following two sides: (1) a new guarded path introduced at a stage shares only its end vertices with any current path, and (2) the end vertices of a newly introduced guarded path are on or very close to some *outer cycle*, whose all vertices belong to the infinite face of the robber territory. All the guarded paths in our strategy are chosen so that they are almost distinct, excluding their end vertices and a special situation in which two new paths are simultaneously introduced at a stage. A strategy with capture time less than $2n$ can then be obtained. The capture time using our strategy is provably faster than $2n$.

II. Main Result

We first introduce a new concept used in our strategy. The *outer cycles* of a graph are created to choose from them the end vertices of guarded paths.

Definition 1: The subgraph $C(R_i)$ of R_i is defined as the set of the outer cycles, whose all the vertices and edges belong to the *infinite* (exterior) face of R_i (Fig.1). In the case of a polyhedral graph, where all of its faces can be considered as interior ones, we can choose any face as the infinite face. For graphs with multiple planar embeddings, outer cycles are made from the infinite face of the planar straight line drawing.

Before we begin, there are two propositions which we use throughout our capture strategy.

Proposition 1: At the end of each stage i, we have at least one free cop.

Proposition 2: During our strategy, any guarded path introduced at stage i shares only its end vertices with each of the current paths. For the two guarded paths introduced at the same stage, they have a common end vertex, and may share a subpath starting from that common vertex.

Suppose that at the beginning of stage i, one or two current paths are guarded by the cops so as to prevent the robber from leaving R_i. These paths will be denoted by P_i^1 and P_i^2. Since P_i^1 and P_i^2 are assigned at the end of R_{i-1}, $P_i^1 \cap R_i = \emptyset$ and $P_i^2 \cap R_i = \emptyset$. We assume that P_i^1 always exists, i.e., R_i $(i > 0)$

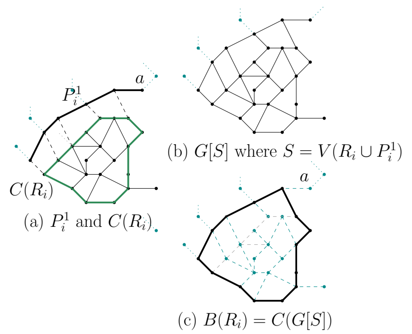

Fig. 1. In (a) the current guarded path P_i^1 is shown in thick black line, and the outer cycle of R_i ($C(R_i)$) is shown in thick shaded cycle. (b) shows the subgraph induced on G by the vertex set $V(R_i \cup P_i^1)$, drawn in black lines and dots. The thick cycle in (c) represents the graph $B(R_i) = C(G[V(R_i \cup P_i^1)])$.

can NEVER be assigned without P_i^1. The newly introduced path(s) at stage i will be denoted by P or/and Q.

For two end vertices of a new guarded path, one may consider to choose them from $C(R_i)$. But, it is difficult or even impossible in some cases for the new path to share a common vertex with P_i^1 or P_i^2. To overcome this difficulty, we will select the end vertices from the outer cycle of the subgraph induced by the vertices of the union of R_i, P_i^1 and P_i^2.

Definition 2: Let $S = V(R_i \cup P_i^1 \cup P_i^2)$. The graph $B(R_i)$ (of enlarged outer cycles) is defined as $C(G[S])$.

Note that $B(R_i)$ consists of only cycles, and thus some vertices of R_i, P_i^1 and P_i^2 may not belong to $B(R_i)$. For instance, the vertex a of P_i^1 (Fig.1(a)) does not belong to $B(R_i)$. See Fig.1(c).

The capture strategy goes as follows: first we let $R_0 = G$, if $B(R_0) = \emptyset$, we simply place our cops somewhere and chase after the robber since G is a tree. Otherwise, we first place three cops in a vertex e_0 of $B(R_0)$. At this point, stage 0 ends and $R_1 = R_0 - e_0$.

In our refined strategy, we utilize up to two cops simultaneously in a stage. In such a case, that stage ends when both cops control two different guarded paths introduced in that stage. The two paths start from a common vertex u, and the other end vertices v_1 and v_2 (of path P and Q, respectively) are chosen from a cycle $C \subseteq B(R_i)$ such that they are in R_i. This can simply be done by considering a vertex $\lfloor \frac{|C|}{3} \rfloor$ distance away from u along C, and if that vertex is not in R_i (belongs to P_i^1), we repeat considering the next vertex away from u. Then, we create shortest paths $P = \pi(u, v_1)$ and $Q = \pi(u, v_2)$ on the subgraph induced by the vertices of R_i and u.

For R_1, the current path $P_1^1 = e_0$. We let $u \leftarrow e_0$ and $C \leftarrow C_0$ where $C_0 \subseteq B(R_1)$ is the cycle containing e_0. Two cops now controls shortest paths P and Q, and the robber territory R_2 is the component containing the robber r in subgraph $R_1 - (P \cup Q)$.

The robber territory R_i ($i \geq 2$) has three different occurrences: (a) P_i^1 has at most one vertex whose neighbor(s) is in R_i, (b) P_i^1 has two or more vertices whose neighbors are in R_i and $P_i^2 = \emptyset$, and (c) P_i^1 has two or more vertices whose neighbors are in R_i and $P_i^2 \neq \emptyset$. We reduce R_i recursively using case-analysis until R_{i+1} is a tree, which is the base case. In (a) and (b), we use method described above for two cops simultaneously controlling two different paths. For case (c), we need yet another concept called *active path*.

Definition 3: Active Path: Let $P = \pi(a, b)$ be a current guarded path at stage i, m (n) the first vertex of P from a (b) that has a neighbor in R_i. The subpath $P(m, n)$, from m to n, is called an active path of P.

Lemma 1: Suppose path P is guarded by some cop c. Then it suffices for c to guard the active subpath of P.

By Proposition 2, P_i^1 and P_i^2 has a common vertex e_i. For case (c), we construct P by choosing one end vertex x from the end vertex of active path of P_i^2 closest to e_i, and another end vertex y from the end vertex of active of P_i^1 farthest from e_i.

In evaluating the capture time, we discard the movements of the cops controlling a paths. This is due to the fact that only the movements to control the paths affect the length of a stage, as the stage ends when the cops successfully control paths introduced at that stage.

Each guarded path in our strategy is traversed by the cops in three different occurrences: (1) when first introduced (to be controlled by a cop), (2) after making another current path obsolete (being intermediate path between obsolete path and newly introduced path), and (3) after becoming obsolete (the cop leaves the path to control another new path).

Lemma 2: In our capture strategy, each guarded path is traversed no more than twice of its length.

Proof. A life of a path $U = \pi(p, q)$ (with active subpath $U(r, s)$) used in our strategy is traversed, over multiple stages (from its introduction until obsolescence), with a length at most $2|U|$.□

Theorem 1: In our capture strategy, all the paths used in evaluating the lengths of stages are distinct, excluding their end vertices.

Proof. Supposed the guarded paths P_i and P_{i+1} are introduced during stage i and stage $i+1$, respectively. It follows from Theorem 3 that, excluding their end vertices, P_i is distinct from P_{i+1}. When two paths are introduced at the same stage and they may share a subpath (case (b)), only one of them (i.e., the one traversed by the free cop whose number of movements is larger) is used to evaluate the length of that stage. Thus, all the guarded paths used in evaluating the stages' lengths are then distinct, excluding their end vertices. □

Theorem 2: For the cops and robbers game on a planar graph G of n vertices with three cops, $capt_3(G) \leq 2n$.

Proof. The theorem directly follows from Theorem 4 and Lemma 5. □

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On the Sigma Coloring of Quartic Circulant Graphs

Paul Adrian D. Luzon, Mari-Jo P. Ruiz, Mark Anthony C. Tolentino[1]
paulluzon48@yahoo.com,mruiz@ateneo.edu,mtolentino@ateneo.edu
Ateneo de Manila University

In [2], G. Chartrand, F. Okamoto, and P. Zhang defined the concept of the *sigma chromatic number* of a graph as follows: For a non-trivial connected graph G, let $c : V(G) \to \mathbb{N}$ be a vertex coloring of G. For each $v \in V(G)$, let $N(v)$ denote the *neighborhood of* v, i.e., the set of vertices adjacent to v. Moreover, the *color sum* of v, denoted by $\sigma(v)$, is defined to be the sum of the colors of the vertices in $N(v)$. If $\sigma(u) \neq \sigma(v)$ for every two adjacent $u, v \in V(G)$, then c is called a *sigma coloring* of G. The minimum number of colors required in a sigma coloring of G is called its sigma chromatic number and is denoted by $\sigma(G)$. In the following example, the colors are placed inside the vertices while the color sums are in square brackets outside.

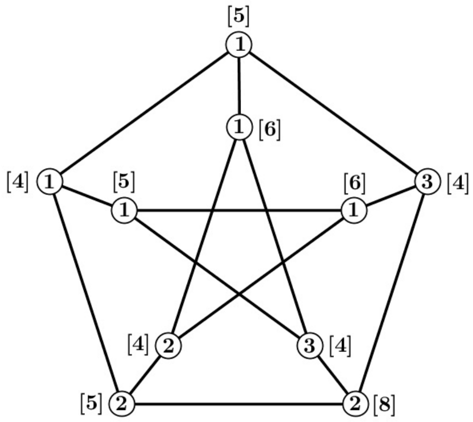

Figure 1: An example of a sigma coloring of a graph.

In the same paper [2], the sigma chromatic numbers of complete multipartite and other standard graphs are determined. In a more recent paper [3], A. Dehghan, M. Sadeghi, and A. Ahadi prove that for each positive integer k, the problem of deciding whether the sigma chromatic number of a 3-reqular graph equals k is **NP**-complete.

Let S be a subset of $\mathbb{Z}_n \setminus \{0\}$ with the property $a \not\equiv \pm b \pmod{n}$ for any two distinct elements a, b of S. The circulant graph $C_n(S)$ is defined to be the graph with vertex set $\mathbb{Z}_n$ such that each v is adjacent to $(v \pm a) \bmod n$ for each $a \in S$.

In our previous work [5], we determined the sigma chromatic number of certain families of circulant graphs, namely, $C_n(1,2)$, $C_n(1,3)$, and $C_{2n}(1,n)$. The result on $C_n(1,2)$ is as follows:

Result 1. *Let $n \geq 6$ be an integer. Then*

$$\sigma(C_n(1,2)) = \begin{cases} 2, & n = 6k, k \in \mathbb{N}, \\ 3, & \textit{otherwise}. \end{cases}$$

In this work, we continue our investigation of the sigma chromatic number of more general classes of circulant graphs. In particular, result 1 extends to the following:

Result 2. *Let $n \geq 6$ be an integer, $a = 1, 2, ..., \lfloor n/2 \rfloor$, and set $d = gcd(n, a)$. Then*

$$\sigma(C_n(a, 2a)) = \sigma\left(C_{n/d}(1,2)\right).$$

[1]speaker

Result 2 allows us to establish the following result, which we have previously conjectured:

Result 3. *Let $n \geq 6$ and $D = \{a, b\}$ be a generating subset of $\mathbb{Z}_n$ not containing 0. Then any connected circulant graph $C_n(D)$ is sigma 3-colorable.*

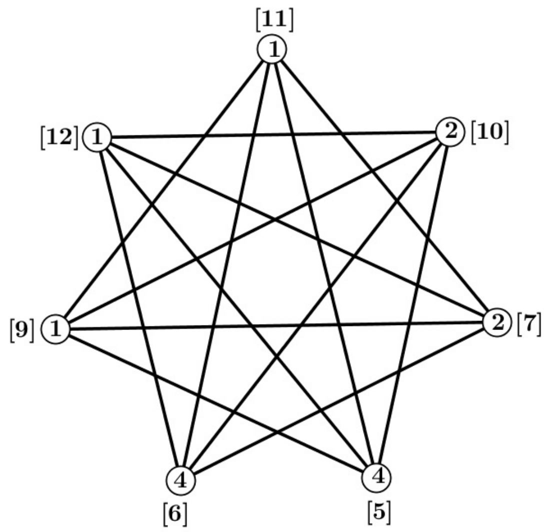

Figure 2: An optimal sigma coloring of $C_7(2, 4)$.

We also present the following result for disconnected circulant graphs of degree at most four.

Result 4. *Let $n \geq 6$ and $D = \{a, b\}$ be a subset of $\mathbb{Z}_n$ not containing 0. If D is not a generating subset of $\mathbb{Z}_n$, then $C_n(D)$ is disconnected and its components are isomorphic circulant graphs of smaller order. Moreover, $C_n(D)$ is sigma 3-colorable provided that its components are not isomorphic to K_5.*

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A construction of tournaments satisfying some adjacency properties

SHOHEI SATAKE (Nagoya University)

1 Introduction

Let n and k be positive integers. A tournament T is called *n-existentially closed* (n-e.c.) if for any disjoint two subsets of vertices, $A, B \subset V(T)$ with $|A \cup B| = n$, there exists $z \notin A \cup B$ which directs to all vertices in A and is directed from all of B. And for a prime power $q \equiv 3 \pmod 4$ and the finite field $\mathbb{F}_q$, the *Paley tournament* T_q is the tournament with $V(T_q) = \mathbb{F}_q$ and $E(T_q) = \{(x, y) \in \mathbb{F}_q^2 \mid x - y \text{ is quadratic}\}$.

From the work in Erdős-Rényi [3], finite random tournaments are almost surely n-e.c. After this work, some explicit constructions of such tournaments have been investigated. Especially, it is well-known that sufficiently large Paley tournaments are n-e.c. for all n (cf. [5]). The property of n-existentially closedness is applied to the theory of 0-1 law and network theory (cf. [1], [2]).

On the other hand, similar adjacency properties also have been investigated. For example, Schütte raised the problem of the tournaments satisfying the property denoted S_k. A tournament T satisfies S_k if for $A \subset V(T)$ with $|A| = k$, there exists $z \notin A$ which directs to all vertices in A (see also [1]). Erdős [4] showed the existence of such tournaments by a probabilistic method. As an explicit construction, it is also well-known that sufficiently large Paley tournaments satisfy S_k for all k (cf. [5]).

2 Our results

In this talk, we define some "Paley-like" tournaments. First, let $q \equiv 5 \pmod 8$ be a prime power and $\chi(x) = \exp(\frac{2\pi i}{4} t)$ where g is a generator of the mutiplicative group $\mathbb{F}_q^*$ and $x = g^t$. We define the tournament $T_q^{(4)}$ by setting $V(T_q^{(4)}) = \mathbb{F}_q$ and $E(T_q^{(4)}) = \{(x, y) \in \mathbb{F}_q^2 \mid \chi(x-y) = 1, i\}$. This is well-defined since $\chi(-1) = -1$ and χ takes only ± 1, $\pm i$. Then we obtain the following results.

Theorem 1 *If $q > n^2 2^{3n-1}$, $T_q^{(4)}$ is n-e.c.*

Theorem 2 *If $q > 2(k-1)^2\{(2+\sqrt{2})^k - 1\}^2$, $T_q^{(4)}$ satisfies S_k.*

Similarly, we also define the following tournament. Let $q \equiv 7 \pmod{12}$ be a prime power and $\mu(x) = \exp(\frac{2\pi i}{6} t)$ where h is a generator of the mutiplicative group $\mathbb{F}_q^*$ and $x = h^t$. We define the tournament $T_q^{(6)}$ by setting $V(T_q^{(6)}) = \mathbb{F}_q$ and $E(T_q^{(6)}) = \{(x, y) \in \mathbb{F}_q^2 \mid \mu(x - y) = 1, \exp(\frac{\pi i}{3}), \exp(\frac{2\pi i}{3})\}$. This is well-defined since $\mu(-1) = -1$ and μ takes only ± 1, $\pm\exp(\frac{\pi i}{3})$, $\pm\exp(\frac{2\pi i}{3})$. Then we can also show that $T_q^{(6)}$ is n-e.c. and satisfies S_k for sufficiently large q.

Theorem 3 *If $q > n^2 2^{6n-4}$, $T_q^{(6)}$ is n-e.c.*

Theorem 4 *If $q > 2(k-1)^2(2^{2k} - 1)^2$, $T_q^{(6)}$ satisfies S_k.*

If possible, we will also discuss about the tournaments with small orders and some related topics.

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The p-center problem with centers constrained to two perpendicular lines

Binay Bhattacharya,[1] Ante Ćustić,[1] Sandip Das,[2] Yuya Higashikawa,[3] Tiko Kameda,[1] and Naoki Katoh[4]
([1]Simon Fraser University, Canada, [2]Indian Stat. Inst., Kolkata, India,
[3]Chuo University, Japan, [4]Kwansei University, Japan)

I. Introduction and summary

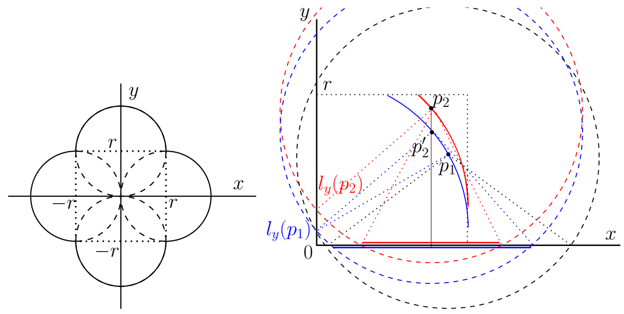
Let $P = \{p_1, p_2, \ldots, p_n\}$ be a set of unweighted points in the x-y plane. For the p-center problem with the centers constrained to a single line, it is known that the *weighted* version can also be solved in $O(n \log n)$ time [6]. When centers are constrained to two parallel lines, we have recently shown that the weighted p-center problem can be solved in $O(n \log^2 n)$ time [1].

We investigate the p-center problem, where the centers are constrained to the x- and y-axes. A point is said to be r-*covered* by a center c, if it is within distance r from c. A problem instance is said to be (r, p)-*feasible*, if each point is r-covered by at least one of the p centers. After preprocessing, which takes $O(n \log n)$ time, we show that (r, p)-feasibility can be tested in linear time. Then, using parametric search and a sorting network [2], [5], we can solve the p-center problem under consideration in $O(n \log n)$ time, improving the previously best $O(n \log^2 n)$ time algorithm [1]. We presort the points in P according to their x- and y-coordinates separately, but we cannot afford to sort the points on the x- and y-axes at distance r from all the points for each value of r, since it would require too much time. This is the main challenge.

II. Approach

Let X (resp. Y) denote the x-axis (resp. y-axis). To be covered by circles of radius r on the two axes, the given points must be contained in the horizontal or vertical band of width $2r$ defined by two lines, $x = \pm r$ or $y = \pm r$. We first sort the points in P according to both the x- and y-coordinates, separately. Consider the four disks of radius r centered at coordinates $(0, r), (0, -r), (-r, 0)$, and $(r, 0)$, respectively, and let us call $A(r)$ the clover shaped area which is the union of these four disks. See Fig. 1(a).

Clearly, any point outside $A(r)$ must be covered by a circle centered on either the x-axis or y-axis, not both. To find the minimum number q of centers required to cover all the points outside of $A(r)$, we can apply the linear-time 1-line algorithm [3] four times, "outside-in" towards the origin on the two axes, starting from the farthest points in the horizontal and vertical directions, until all the points outside $A(r)$ are covered. Thus the main issue is testing if the set $P(r)$ of points inside $A(r)$, which are not covered by the circles introduced so far, can be covered by $p - q$ centers. Clearly, we need at most four additional centers shown in Fig. 1(a), but we want to find the exact number needed. We thus want to test if they can be covered by one,



(a) (b)

Fig. 1. (a) Area $A(r)$; (b) p_2 dominates p_1.

two or three circles of radius r. By a greedy method [3], we can easily test if all the n points in $P(r)$ can be covered by one, two or three circles on the same axis in $O(n)$ time. Therefore, without loss of generality, we assume that there is one circle, C_y, with center $c_y \in Y$, and the others (one or two) are placed on X. To explain our idea, we first discuss the case where we have just one more circle of radius r, C_x, centered at $c_x \in X$.

For $j = 1, \ldots, 4$ we refer to quadrant i as Q_j in what follows. Let $P_j(r)$ denote the set of points of $P(r)$ that lie in $A(r) \cap Q_j$. The points on the quadrant boundary can be partitioned arbitrarily to make $\{P_j(r)\}$ disjoint. Assume that the points in $P(r)$ can be r-covered by C_y and C_x. For each point $p \in P(r)$, let $l_y(p) \in Y$ (resp. $h_y(p) \in Y$) denote the lowest (resp. highest) position satisfying $d(l_y(p), p) \le r$ (resp. $d(h_y(p), p) \le r$), where $d(a, b)$ denotes the distance between points a and b. We similarly define $l_x(p) \in X$ and $h_x(p) \in X$, where highest (resp. lowest) means the rightmost (resp. leftmost). For each point $p \in P_j(r)$, we define the interval $I_x^j(p) = [l_x(p), h_x(p)] \subset X$. Note that $I_x^j(p)$ may extend to the negative part of X. For two points $p, p' \in P_j(r)$, p' *dominates* p if $I_x^j(p') \subset I_x^j(p)$, and p is *dominated* if there exists a point that dominates it. See Fig. 1(b). Note that "dominates" is a transitive relation. A line segment of length r from point p to the x- or y-axis (i.e., $l_x(p), h_x(p)$, $l_y(p)$, or $h_y(p)$) is called a *radius line*.

Now let us initially place C_y at $c_y = (0, r)$, and move it downwards, trying to cover points in $P_1(r)$. Stop when the upper half-arc of C_y hits a point in $P_1(r)$ for the first time. Fix c_y there, and test if the points in $P_1(r)$ outside C_y can be covered by C_x. As we move C_y downwards along the y-axis, points that were initially in C_y are "released" one by one. Each released point must be covered by C_x. Note that the points $p_i \in P_1(r)$ with $p_i.x > r$ must always be covered by C_x, since C_y cannot cover it.

III. Algorithm

Assume that the points in $P_1(r)$ are renamed $p_1, \ldots, p_m$ and satisfy $p_1.x \geq p_2.x \geq \ldots \geq p_m.x$. The following procedure identifies the highest group of points in $P_1(r)$, whose radius lines to the y-axis cross each other.

Procedure 3.1: `Scan-1L`.[1] Prepare an empty stack S_1.

1) Place p_1 in S_1.
2) For $i = 2, 3, \ldots, m$, carry out the following steps.
   a) If $p_i.y < top(S_1).y$ then do nothing and skip (b) and (c).
   b) While $l_y(p_i) > l_y(top(S_1))$ pop S_1.
   c) $[l_y(p_i) \leq l_y(top(S_1))$ or $S_1 = \emptyset$] Push p_i in S_1.
3) Output S_1 as S_1^*. ■

Lemma 3.2: (a) Any point popped up by Step 2(b) of Procedure `Scan-1L` is dominated by the bottom item in stack S_1^*.
(b) The bottom item of S_1^* is the first point released by C_y. The contents of S_1^* are ordered by $l_y(\cdot)$, with the largest valued item at the bottom.
(c) The right endpoints of the x-intervals of the items in S_1^* are ordered from right to left with the bottom item at the rightmost position. ■

We define a similar procedure `Scan-2R`, which is symmetric to `Scan-1L` with respect to the y-axis, and scans the points in $P_2(r)$ from left to right, using stack S_2. Suppose we have executed `Scan-2R`, which outputs stack S_2^*. Note that the proof of Lemma 3.2 did not assume that $c_x \geq 0$. Therefore, it applies to `Scan-2R`, without assuming $c_x \leq 0$. We can thus conclude that once the point at the bottom of stack S_2^* is covered by a circle on X, we no longer need to worry about the dominated points in $P_2(r)$.

We define `Scan-4L` (resp. `Scan-3R`) for the points in $P_4(r)$ (resp. $P_3(r)$), symmetric to `Scan-1L` (resp. `Scan-2R`) with respect to the x-axis (y-axis). For each i, we work on the points in $P_i(r)$ separately. For example, whenever C_y releases a point in $P_1(r)$ as it moves downward, we update the interval $\mathcal{I}_x^1(r)$, based on the points in S_1^*. This updating requires constant time. Note that the size of $\mathcal{I}_x^1(r)$ is non-increasing as C_y moves downward. Moreover, whenever it shrinks, its right end moves to the left. The left end may or may not move to the right at the same time. Whenever C_y releases a point in S_1^*, we record the updated $\mathcal{I}_x^1(r)$, indexed by the new c_y, in a list of records, R_1. For $j = 2, 3, 4$, define S_j^*, $\mathcal{I}_x^j(r)$, and R_j analogously for the points in $P_j(r)$.

Procedure 3.3: $\mathtt{Q}_i(r)$

1) Initialize $R_i = \Lambda$ (empty list).
2) Run `Scan-`j`X`. (X=L for $j = 1, 4$ and X=R for $j = 2, 3$.)
3) Position C_y so that c_y is at the bottom item in S_j^*.[2]
4) Compute $\mathcal{I}_x^j(r) = \cap_{p \in P_j \setminus C_y} I_x^j(p)$. Record the pair $(c_y, \mathcal{I}_x^j(r))$.
5) While $[c_y$ not at the top of $S_j^*] \wedge [\mathcal{I}_x^j(r) \neq \emptyset]$ do
   (a) Move c_y to the next point in S_j^*, and update $\mathcal{I}_x^j(r)$.
   (b) Append the pair $(c_y, \mathcal{I}_x^j(r))$ to R_j. ■

By Lemma 3.2(c), the size of $\mathcal{I}_x^j(r)$ is non-increasing. We run all the four procedures, $\mathtt{Q}_1, \ldots, \mathtt{Q}_4$. We then merge-sort the records in the resulting $\{R_i \mid j = 1, \ldots, 4\}$, based on the first components (c_y) of the records. These values c_y partition Y into y-intervals, and each such y-interval I_y corresponds to a set of four records, one from each of $\{R_j \mid j = 1, \ldots, 4\}$. Let $\{\mathcal{I}_x^j(r) \mid j = 1, \ldots, 4\}$ be the x-intervals that I_y corresponds to. If $\cap_{j=1}^4 \mathcal{I}_x^j(r)) \neq \emptyset$, then it implies that two circles C_y and C_x such that $c_y \in I_y$ and $c_x \in \cap_{j=1}^4 \mathcal{I}_x^j(r)$ cover all the points in $P(r)$.

If two circles cannot cover all the points in $P(r)$, without loss of generality, we can assume that one circle lies on Y and the other two lie on X. Instead of just a single interval $\mathcal{I}_x^j(r)$, we need to maintain two intervals on X. We have a problem here in that in Step 5(b) of Procedure $\mathtt{Q}_i(r)$ we don't know which of the two intervals should be updated. We solve this problem by making use of a special relation among the items in S_j^*.

Lemma 3.4: We can test (r, p)-feasibility in $O(n)$ time.

We can now apply Megiddo's parametric search [4] with Cole's speed-up [2]. The inputs to the comparators of a sorting network are $\{l_x(p), h_x(p) \mid p \in P\}$.

Theorem 3.5: The (unweighted) p-center problem with the centers constrained to two perpendicular lines can be solved in $O(n \log n)$ time. ■

IV. Conclusion and comments

We presented an $O(n \log n)$ time algorithm for the unweighted p-center problem, where the centers are constrained to the x- or y-axis. If the points are weighted, it appears difficult to solve this problem in time which is a low-degree polynomial in n.

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[1] Quadrant 1 scanned Leftward.

[2] This item has the highest $l_y(\cdot)$ for Scan-1L and Scan-2R, and the lowest $l_y(\cdot)$ for Scan-3R and Scan-4L.

Alternating paths for some bicolored point sets in convex position

Toshinori Sakai*and Jorge Urrutia†

Extended Abstract

Let P be a set of $2n$ points in general position on the plane. Suppose that n elements of P are colored red and n blue. An *alternating path* of P is a simple polygonal whose edges are straight line segments joining pairs of elements of P with different colors. Alternating paths of point sets were first studied by Akiyama and Urrutia [3]. In that paper, an algorithm that decides if an alternating path that covers all the elements of P exists is given when the elements of P are in convex position, i.e. the elements of P are the vertices of a convex polygon.

In [1], the problem of finding an alternating path for a point set in general position is studied. It is shown that if all the red elements of P are separated from all the blue elements by a straight line or if all the blue points are contained in the convex hull of the red points, then there is an alternating path that covers all the elements of P. Later, in [2], this result was used to prove that any point set P in general position always has an alternating path that covers at least half of the elements of P.

We henceforth consider the case where the elements of P are in convex position. Kynčl, Pach and Tóth [5] proved that there exists an alternating path that covers at least $n+\Omega(\sqrt{n/\log n})$ elements of P. This number was improved by Hajnal and Mészáros [4] to $n+\Omega(\sqrt{n})$. As for upper bounds of the number of elements covered by an alternating path, $(4/3)n+O(\sqrt{n})$ is shown, see [2, 4, 5].

Write the elements of P as $p_0, p_1, \ldots, p_{2n-1}$ along the perimeter of the convex hull of P. The indices are to be read modulo $2n$. A subset $R = \{p_i, p_{i+1}, \ldots, p_{i+(r-1)}\}$ is called a *run* if its elements are colored by a same color, and p_{i-1} and p_{i+r} by the other color. The *length* of a run is the number of its elements. P is said to be a *k-configuration* if P consists of k red runs and k blue runs. We show the following results:

Theorem 1 *Let P be a k-configuration and suppose that each run of P has length of a multiple of d. Then P contains an alternating path of length at least* $\dfrac{n}{1-\frac{kd}{2n}}$.

Corollary 1 *Suppose that each run of P has length of a multiple of d, and a longest run has length Md. Then P contains an alternating path of length at least* $\dfrac{n}{1-\frac{1}{2M}}$.

*Department of Mathematics, School of Science, Tokai University, Japan. sakai@tokai-u.jp

†Instituto de Matemáticas, Ciudad Universitaria, Universidad Nacional Autónoma de México, México D.F., México. urrutia@matem.unam.mx

In view of a lower bound $n + k - 1$ shown in [5], we next focus on the case where $k = o(n)$. As one of such a case, we consider the case where all runs of a same color have different length. In this case, we must have $k(k + 1)/2 \leq n$, and hence $k < \sqrt{2n}$.

Theorem 2 *Let r be the ratio of the maximum length of a run to $\sqrt{n}$. Suppose that all runs of a same color have different lengths, and $r < \dfrac{3}{\sqrt{2}} = 2.12....$ Then P contains an alternating path of length at least $\sqrt[3]{\dfrac{9}{2r^2}}\, n - o(\sqrt{n})$.*

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Minimizing the Solid Angle Sum of Orthogonal Polyhedra and Guarding them with $\frac{\pi}{2}$-Edge Guards

I. Aldana-Galván* J.L. Álvarez-Rebollar†
J.C. Catana-Salazar* M. Jiménez-Salinas*
E. Solís-Villarreal* J. Urrutia‡

Abstract

We give a characterization for the orthogonal polyhedron in $\mathbb{R}^3$ that minimizes the sum of its internal solid angles, and prove that their minimum angle sum is $(n-4)\pi$ and their maximum angle sum is $(3n-24)\pi$. We generalize to $\mathbb{R}^3$ the well-known result that in an orthogonal polygon with n vertices, $(n+4)/2$ of them are convex and $(n-4)/2$ of them are reflex. We define a vertex of a polyhedron to be convex on the faces if it is convex or straight in all the faces where it participates, and to be reflex on the faces otherwise. If a polyhedron with n vertices and genus g has k vertices of degree greater than 3 (in its 1-skeleton), we prove that it has $(n+8-8g+3k)/2$ vertices that are convex on the faces and $(n-8+8g-3k)/2$ vertices that are reflex on the faces. Finally, we prove that if the orthogonal polyhedron has k_4 vertices of degree 4, k_6 vertices of degree 6, genus g and h_m holes on its faces, then we can guard it using at most $(11e-k_4-3k_6-12g-24h_m+12)/72$ $\frac{\pi}{2}$-edge guards (i.e., having a visibility angle of $\pi/2$ in the relative interior of each edge), improving the bound given by Viglieta et al for open edge guards.

*Posgrado en Ciencia e Ingeniería de la Computación, Universidad Nacional Autónoma de México, Ciudad de México, México, ialdana@ciencias.unam.mx, {j.catanas, m.jimenez, solis_e}@uxmcc2.iimas.unam.mx

†Posgrado en Ciencias Mateméticas, Universidad Nacional Autónoma de México, Ciudad de México, México, chepomich1306@gmail.com

‡Instituto de Mateméticas, Universidad Nacional Autónoma de México, Ciudad de México, México, urrutia@matem.unam.mx

How to Represent Polytopes

Kengo Nishio and Takehide Miyazaki
National Institute of Advanced Industrial Science & Technology (AIST),
Umezono 1-1-1, Tsukuba, 305-8568, Japan

1. Introduction.

Except for highly symmetric polytopes, there is no method for representing polytopes in general. This causes difficulty in understanding and describing complicated structures. For example, in materials science, the arrangements of atoms in liquids and glasses are often represented as polyhedral tilings. However, there is no method to succinctly describe what polyhedra are tiled in what way. To overcome this problem, we have created a theory for representing polytopes [1,2]. In this presentation, we will present the theory.

Our theory is based on the hierarchy of structures of polytopes: a polyhedron (3-polytope) is a tiling by polygons (2-polytopes), a polychoron (4-polytope) is a tiling by polyhedra, and so on. We first describe what we call the p_3-code. The p_3-code is an algorithm to convert a polyhedron into a p_3-codeword (p_3 for short) that instructs how to construct the polyhedron from its building-block polygons. By generalizing the method, we formulate the p_4-code for polychora. Since the polyhedral tilings of disordered structures are parts of polychora, the p_4-code can be used to describe the arrangements of polyhedra. The theory can be generalized to higher dimensional polytopes.

2. Code for polyhedra.

A polyhedron can be regarded as a tiling by polygons of the surface of a three-dimensional object that is topologically the same as a three-dimensional sphere. We assume that polygons are glued such that (1) any pair of polygons meet only at their sides or corners and that (2) each side of each polygon meets exactly one other polygon along an edge. In this picture, the vertex is a point on a polyhedron at which the corners of polygons meet, and we say that the corners contribute to the vertex. We also say that a polygon (side) contributes to a vertex if one of its corners (endpoints) contributes to the vertex. Similarly, the edge is a line segment on a polyhedron along which the sides of polygons meet. The face of a polyhedron is a polygon.

In our theory, a polyhedron is represented by p_3. The p_3-codeword consists of a polygon-sequence codeword (ps_2) and a side-pairing codeword (sp_2), and is denoted by

$$\begin{aligned} p_3 &= ps_2; sp_2 \\ &= p_2(1)p_2(2)p_2(3)\cdots p_2(N_{\text{polygon}}); y(1)x(1)y(2)x(2)y(3)x(3)\cdots y(N_{\text{na-pair}})x(N_{\text{na-pair}}). \end{aligned}$$

Here, $p_2(i)$ is the number of sides on the polygon i. N_{polygon} is the number of polygons on the polyhedron. ";" is a separator. The pair of $y(i)$ and $x(i)$ is what we call a non-curable additional pair (na-pair). By the na-pair $y(i)x(i)$, we mean that the sides $y(i)$ and $x(i)$ should be glued together.

A triangle prism (Fig 1), for example, is represented by $p_3 = ps_2 = 34443$. As with a triangle prism, many polyhedra can be represented by only ps_2s. But the other polyhedra need sp_2s.

3. Code for polychora.

A polychoron can be regarded as a tiling by polyhedra of the surface of a four-dimensional object that is topologically the same as a four-dimensional sphere. We assume that polyhedra are glued together such that (1) any pair of polyhedra meet only at their faces, edges, or vertices and that (2) each face of each polyhedron meets exactly one other polyhedron along a ridge. The 0-face, peak, and ridge are a point, line segment, and area on a polychoron, where the vertices, edges, and faces of polyhedra meet, respectively. The cell of a polychoron is a polyhedron.

A polyhedron is represented by p_4. The p_4-codeword consists of a polyhedron-sequence codeword (ps_3) and a face-pairing codeword (fp_2), and is denoted by

$$p_3 = ps_3; fp_2$$
$$= p_3(1)p_3(2)p_3(3)\cdots p_3(N_{\mathrm{polyhedron}}); w(1)z(1)v(1)\cdots w(N_{\mathrm{na-pair}})z(N_{\mathrm{na-pair}})v(N_{\mathrm{na-pair}}).$$

Here, $p_3(i)$ is the p_3-codeword of the polyhedron i. $N_{\mathrm{polyhedron}}$ is the number of polyhedrons on the polychoron. ";" is a separator. The pair of $w(i)$, $z(i)$ and $v(i)$ is an na-pair. By the na-pair $w(i)z(i)v(i)$, we mean that the faces $w(i)$ and $v(i)$ should be glued together in such a way that the edge $z(i)$ of the face $w(i)$ is glued to the smallest-ID edge of the face $v(i)$.

A polychoron composed of two 3333-polyhedra and four 34443-polyhedra (Fig. 2), for example, is represented by $p_4 = ps_3 = 3333\ 34443\ 34443\ 34443\ 34443\ 3333$. As with this polychoron, many polychora can be represented by only ps_3s. But the other polychora need fp_2s.

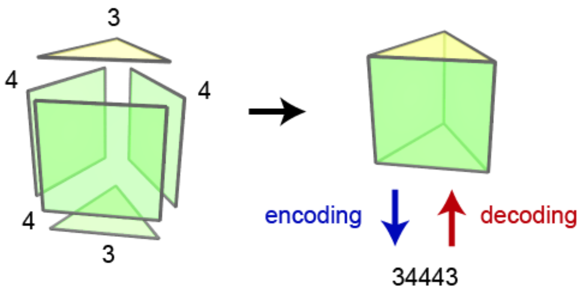

Figure 1. Example of p_3.

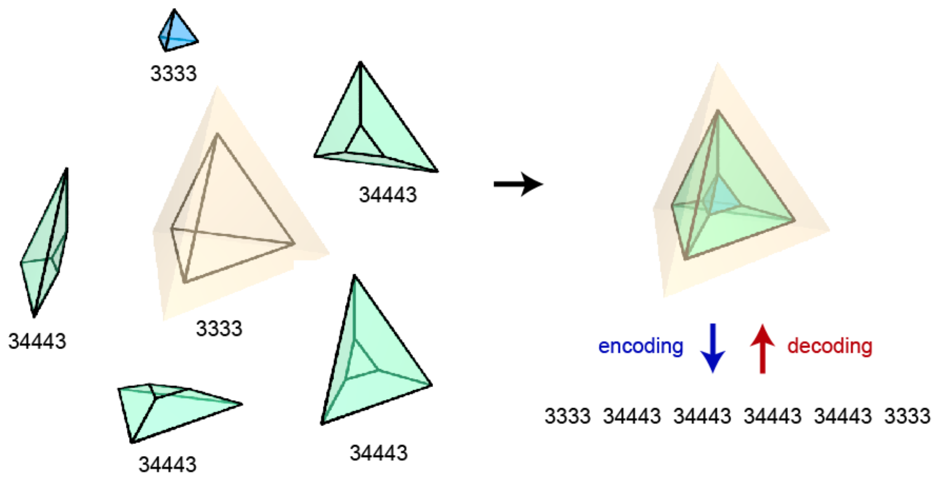

Figure 2. Example of p_4.

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Selecting K Points that Maximize the Convex Hull Volume

Günter Rote, Freie Universität Berlin
(joint work with Kevin Buchin, Karl Bringmann, Sergio Cabello, and Michael Emmerich)

Problem Statement and Result. We are given a set S of n points in three dimensions and an integer K. We are looking for a subset of at most K points whose convex hull has the largest possible volume. We show that this problem is W[1]-hard, by reduction from Grid Tiling.

Motivation. We think of S as a large set of options from which a user has to make a selection. For example, we might have a decision problem with complicated side constraints with *three* objectives for which the trade-off and the relative importance has not been specified in advance. The set S of feasible objective function triples might be the result of an enumeration algorithm or a simulation that tries out various scenarios. In the end, we want to present a variety of options to the decision-maker to make the choice, based on the available options and on personal preferences. It is of course humanly impossible to decide among thousands of options. Therefore we would like to reduce the set S to a small *representative* subset of at most K options [3].

There are many quality measures for such a selection that one could consider, for example diversity, measured in terms of pairwise distances [1]. We choose a very simple criterion, namely the volume of the convex hull of the selected points. Under this criterion, only points on the boundary of the convex hull are selected.

Our result implies, under the standard complexity-theoretic assumption that W[1] $\neq$ FPT, that the problem cannot be solved in $f(K)n^C$ time, for any function f and any constant C. In other words, the problem is not fixed-parameter tractable. Under the Exponential Time Hypothesis (ETH), our reduction even rules out an $n^{o(\sqrt{K})}$ algorithm.

Related Results. In two dimensions, the problem can be solved by straightforward dynamic programming in $O(n^3K)$ time. As for approximating the maximum volume, it is not hard to design a polynomial-time approximation scheme, based on the notion of core-sets.

In multicriteria optimization, it is customary to consider the *dominated* volume of S. (A point $x \in \mathbb{R}^d$ is dominated by some point $s \in S$ if it is less than or equal to s in all coordinates.) For the problem of selecting K points that maximize the dominated volume in the nonnegative orthant, we could show NP-hardness by a reduction from independent set in planar graphs of maximum degree 3. These results are joint work with Kevin Buchin, Karl Bringmann, Sergio Cabello, and Michael Emmerich.

The Reduction. As mentioned in the beginning, we will reduce the Grid Tiling [2] to our volume maximization problem. This problem is W[1]-hard, because it contains the k-Clique problem in graphs as a special case. For a parameter k, this problem is defined as follows. (See the figure for a pictorial illustration.)

> INPUT: $k \times k$ subsets $C_{ij} \subseteq \{1, \dots, n\}^2$, $1 \leq i, j \leq k$.
> QUESTION: Are there functions $u, v\colon \{1, \dots, k\} \to \{1, \dots, n\}$, such that $(u(i), v(j)) \in C_{ij}$ for all $1 \leq i, j \leq k$?

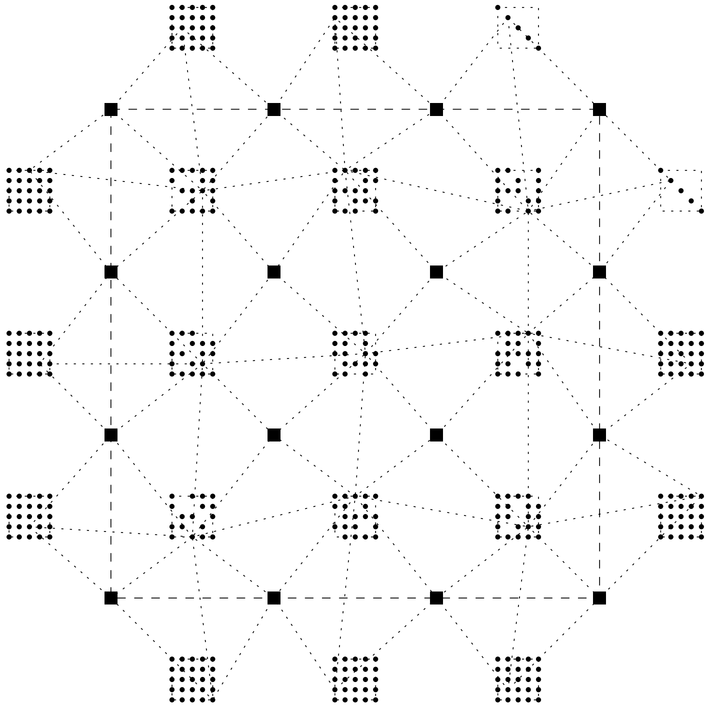
The figure shows the situation $k = 3$ and $n = 5$. The points of the $9 = 3 \times 3$ small square patches inside the large dashed square represent the subsets C_{ij}. The square patches are horizontally and vertically connected. The Grid Tiling problem asks for a selection of one point from each patch such that the points in horizontally adjacent patches have the same horizontal offset from the center of the patch and the points in vertically adjacent patches have the same vertical offset from the center.

The reduction will generate $N = O(n^2k^2)$ points, and the number of points to be selected is $K := 16k^2 + 8k + 5 = O(k^2)$. The figure shows a top view from the point set.

The points S will lie near the concave paraboloid surface

$$z = f(x, y) = -12(x^2 + y^2).$$

The centers of the square patches are at coordinates $i, j \in \{1, \dots, k\}$ and the points in the patches are offset by x_i, y_i such that the selected points can be written as $(i + x_i, j + y_j)$. We



set their z-coordinate to $f(i + x_i, j + y_j) + 6(x_i^2 + y_i^2)$, i.e., these points are slightly above the paraboloid, but still in convex position. In addition we have fixed *anchor* points (shown as boxes) at half-integral positions $(i + \frac{1}{2}, j + \frac{1}{2}, f(i + \frac{1}{2}, j + \frac{1}{2}) - 3)$, below the paraboloid. These points ensure that the combinatorics of the upper convex hull will look as in the figure. If one works out the volume of the convex hull, it can be expressed as

$$\text{const} - \sum 4(x_{ij} - x_{i+1,j})^2 - \sum 4(y_{ij} - y_{i,j+1})^2 + \text{lower-order terms},$$

where the two sums are over all horizontally adjacent patches and all vertically adjacent patches. If the square patches are small enough (much smaller than in the drawing), the quadratic terms in the formula dominate, and the maximum volume can be obtained only if the x_{ij} in each row are all equal and the y_{ij} in each column are equal, i.e., the Grid Tiling Problem has a solution.

We make the whole area finite by continuing the horizontal and vertical "tracks" outside the figure and reflecting them at diagonal boundaries, thus turning them into closed tracks. (A small section of such a diagonal boundary appears in the upper-right corner of the figure.) The whole point set lies over a tilted square domain with corners $(-2k + \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, -2k + \frac{1}{2})$, $(2k + \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 2k + \frac{1}{2})$. To close the body off, we add 4 points below the corners of the square.

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Laguerre Voronoi Diagram as a Tool for Fitting Spherical Tessellations Using Planar Photographic Images

Supanut Chaidee[1] and Kokichi Sugihara[2]

Graduate School of Advanced Mathematical Sciences, Meiji University
4-21-1 Nakano, Nakano-ku, Tokyo 164-8525, Japan
[1]schaidee@meiji.ac.jp, [2]kokichis@meiji.ac.jp

Abstract

We propose a method for fitting the spherical Laguerre Voronoi diagrams to spherical tessellations which are acquired from planar photographic images. The framework is applied using the properties of a polyhedron corresponding to the spherical Laguerre Voronoi diagram. The experiments were performed using fruit skin photos.

1 Introduction

Many natural phenomena display as spherical tessellations. From mathematical viewpoint, the Voronoi diagram can be used as a tool for modeling the pattern formation if the pattern is close to the Voronoi diagram.

There are various generalizations of the Voronoi diagram. One of the generalization is weighted Voronoi diagram. However, most of the weighted Voronoi diagrams contain complicated curved Voronoi edges. We mainly focus on the Laguerre Voronoi diagram (or power diagram) [1, 6] whose Voronoi edges are straight lines.

To model the spherical object, the Laguerre Voronoi diagram on the sphere should be concerned. The concept and properties of spherical Laguerre Voronoi diagram (SLVD) were investigated in [8] by defining the spherical circle $\tilde{c}_i = \{p \in U | \tilde{d}(p_i, p) = r_i\}$ on the sphere U where p_i and r_i are the center and radius of circle $\tilde{c}_i$, respectively. The Laguerre proximity is defined by $\tilde{d}_L(p, \tilde{c}_i) = \cos \tilde{d}(p, p_i)/\cos r_i$. Recently, we [3] investigated the properties of polyhedra corresponding to the SLVD and solved the SLVD recognition problem.

From an application viewpoint, we studied a class of spherical tessellations called spike-containing objects, the spherical tessellations containing spike dots which are regarded as Voronoi generators. We used the SLVD for fitting tessellations from spike-containing object photos [2]. Using the properties of a polyhedron corresponding to the SLVD in [3], we generalized the object to the spherical tessellation object, which does not necessarily contain generators in the tessellation, and gave a framework for approximating a given tessellation using the SLVD in [4].

In this study, we propose a framework for fitting planar photographic images using the SLVDs since it is not easy to get a spherical tessellation data from a real world object. The approximation method in [4] is applied to the data from photos. The adjustment of generating circles is considered to pursue the real world assumptions. Finally, we conduct experiments using photographic images of fruit skins to check the validity of our method.

2 Prelimalinaries

Assume that a tessellation and an SLVD are on the unit sphere U in $\mathbb{R}^3$, where the center of the sphere is located at the origin $O(0, 0, 0)$ of the Cartesian coordinate system. The tessellation $\mathcal{T} = \{T_1, ..., T_n\}$ consists of 3-regular convex spherical polygons.

2.1 Theoretical aspects of SLVD

From the SLVD construction algorithm in [8], the following fact is directly implied.

Proposition 1. *$\mathcal{L}$ is SLVD if and only if there is a convex polyhedron $\mathcal{P}$ containing the center of the sphere whose central projection coincides with $\mathcal{L}$.*

This fact is used for the SLVD recognition problem in [3]. In brief, starting from an arbitrary tessellation vertex v which is adjacent to sites i, j, k, we firstly construct the plane P_i of ith site and the plane $P_{i,j}$ passing through geodesic arc separating cell i and j. The second plane P_j is constructed as the plane passing through the intersection of P_i and $P_{i,j}$ and one more point on the jth side. The third plane is constructed from the intersection of P_i and $P_{i,k}$, and P_j and $P_{j,k}$. After that, we can generate all planes corresponding to each generator.

The following theorem confirms that a polyhedron of the SLVD is unique up to any first pair of planes.

Theorem 1. [3] There are exactly four degrees of freedom of the construction of planes composing a polyhedron $\mathcal{P}$ with respect to the given SLVD.

2.2 Modeling assumption

We define a *spherical tessellation object* as an object which can be approximated as 3-regular convex tessellation on the sphere. If there is a point of degree $k+2 \geq 3$, we separate the point to k points of degree 3 with an infinitesimally small distance $\epsilon > 0$ of k points in such a way that the tessellation is still a convex tessellation.

If the given $\mathcal{T}$ is not exactly the SLVD, the difference between tessellation $\mathcal{T}$ and the SLVD $\mathcal{L}$ occurs.

For each corresponding cell T_i and L_i of ith site, let $A_i = T_i \cap L_i$ be a spherical polygon and $A_\mathcal{T}, A_\mathcal{L}$ be the areas of tessellation $\mathcal{T}, \mathcal{L}$, respectively. Let $A = \sum_{i=1}^{n} \text{Area}(A_i)$, where $\text{Area}(A_i) = 0$ if $A_i = \emptyset$. The discrepancy, the difference of two corresponding tessellations, is defined by

$$\Delta_{\mathcal{T},\mathcal{L}} = \frac{(A_\mathcal{T} - A) + (A_\mathcal{L} - A)}{A_\mathcal{T} + A_\mathcal{L}}.$$

3 Main Framework

We overview the framework to fit the given spherical tessellation $\mathcal{T}$ using the SLVD which was proposed in [4]. We firstly construct an SLVD $\mathcal{L}$ corresponding to $\mathcal{T}$ using polyhedron construction algorithms in [3]. Since each face of the polyhedron corresponds to a plane, we adjust planes in such a way that the discrepancy of $\mathcal{T}$ and $\mathcal{L}$ converges to a local minimum. The plane adjustment process is done by minimizing the discrepancy function with respect to plane parameters $\cup\{A_i, B_i, C_i\}$ of plane equations $A_i x + B_i y + C_i z = 1$ for all i. The optimization is performed using the Nelder-Mead method.

3.1 Retriving information from photos

We firstly take a photo from a spherical tessellation object. A 3-regular convex tessellation on the plane is extracted from the photo. Then we project the tessellation onto the upper hemisphere. Since the information acquired from the photographic image is only a part of the hemisphere, the proposed framework in [4] should be modified as follows.

From the photo, we extract the tessellation not only tessellation boundaries but also edges emanating from the tessellation vertices which are the outermost of the tessellation. These edges are called as *branches* of the tessellation. Tessellation itself and branches are projected onto the unit sphere. For some cell l at the boundary area of tessellation, we assume that the cell l is adjacent to a cell $n+p$ for some p which is unbounded polygonal domain surrounded by edges of the cell l and some branches. We apply a framework in [4] to these planes information for constructing a polyhedron.

Since the data from a photograph is a part of the hemisphere, one more plane which bounds the intersections of halfspaces should be concerned. If the extracted planar tessellation is projected to the upper hemisphere, we construct a *closed plane* $z = c$ for some small $c < 0$. Therefore, the number of constructed plane is $n+q+1$, where q is the number of the unbounded polygonal domains. Finally, we are able to apply the framework in [4] as well.

Remark that the computation of discrepancy as defined in section 2.2 is only performed for n cells although the SLVD is constructed for $n+q+1$ cells.

3.2 Adjustment of generator position

From the proposed framework, we can find the best fit SLVD which fits a given tessellation. However, it is preferable to choose the generators set whose generator positions lay inside tessellation cells, and each generator is close to the centroid of each cell. Besides, the radius of each circle should be a positive number. These assumptions are settled by observing the real world pattern formations which were reviewed by [5].

Since the best fit SLVD is acquired from the provided framework, we adjust the generator positions following to the mentioned assumption. The adjustment of generators affects the change of polyhedron, and we desire a polyhedron whose projection is the same SLVD. Therefore, we use Theorem 1 as a key for this adjustment.

The choice of the four degrees of freedom is performed as follows. Suppose that the first pair of cells is i and j. Without loss of generality, let $X(x, y, z) \in \mathbb{R}^3$ be a point for constructing plane P_i. The choice of first three degrees of freedom comes from the construction of a plane whose a normal vector and the distance between plane and origin point are $\mathbf{X}$ and $\|\mathbf{X}\|$, respectively. The fourth degree of freedom is the angle θ between the plane $P_{i,j}$ and the plane P_j measured from $P_{i,j}$ to P_j anticlockwisely. With these four degrees of freedom, we can construct a polyhedron uniquely.

Since each plane $a_i x + b_i y + c_i z = 1$ of ith cell corresponds to the generator position $p_i := (a_i, b_i, c_i)/\|(a_i, b_i, c_i)\|$, the distance between p_i and the centroid c_i of the SLVD ith cell can be measured using the geodesic distance, say $\tilde{d}(p_i, c_i)$. For all $i = 1, ..., n$, we define the objective function $f(x, y, z, \theta) := \sum_{i=1}^{n}(\tilde{d}(p_i, c_i))^2$ and minimize $f(x, y, z, \theta)$. The minimization of this objective function is executed using Nelder-Mead method.

To hold the positive radii assumption, we refer the generalization of spherical circle radius defined by [3]. Therefore, we shrink the polyhedron toward the origin in such a way that all radii are real numbers.

4 Concluding Remarks

We performed the experiments using the photographic images of sugar apple and raspberry which are classified as spherical tessellation object. From the experiments, the tessellations were well-fitted, and those generator positions are adjusted to the location where is close to the centroid. Therefore, this framework can be applied to the real world patterns. It is promising that the a SLVD can be used as a tool for constructing mathematical models of polygonal pattern formation.

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Depth first search in claw-free graphs

Extended abstract

Gábor Wiener*
Department of Computer Science and Information Theory
Budapest University of Technology and Economics
e-mail: wiener@cs.bme.hu

All graphs in this paper are simple, finite, and undirected; the vertex set of a graph G is denoted by $V(G)$. A graph is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. A graph G is *traceable* if it contains a hamiltonian path. The *minimum leaf number* $\mathrm{ml}(G)$ is the minimum number of leaves (vertices of degree 1) of the spanning trees of G. The *minimum branch number* $\mathrm{s}(G)$ is the minimum number of branches (vertices of degree at least 3) of the spanning trees of G. A tree T is a k*-tree* if all vertices have degree at most k. The minimum degree of G is denoted by $\delta(G)$ and the minimum sum of degrees of k independent vertices of G is denoted by $\delta_k(G)$. The *depth first search (DFS)* of a connected graph G (see e.g. [4]) produces a spanning tree of G, called a DFS-tree, rooted at some node r; the leaves of a DFS-tree different from r will be called *d-leaves* of the DFS-tree.

Hamiltonian properties of claw-free graphs have been examined for more than three decades; one of the early results is due to Matthews and Sumner [6] and was also found independently by Liu, Tian, and Wu [5].

Theorem 1. (Matthews and Sumner, Liu et al., 1985) *Let G be a connected claw-free graph of order n. If $\delta_3(G) \geq n-2$, then G is traceable.*

Gargano, Hammar, Hell, Stacho, and Vaccaro [2] proved a generalization of Theorem 1 concerning the minimum branch number.

Theorem 2. (Gargano et al., 2002) *Let G be a connected claw-free graph of order n and let k be a nonnegative integer. If $\delta_{k+3}(G) \geq n-k-2$, then $\mathrm{s}(G) \leq k$.*

This result was generalized further by Salamon [7].

Theorem 3. (Salamon, 2010) *Let G be a connected claw-free graph of order n and let k be a nonnegative integer. If $\delta_{k+1}(G) \geq n-k$, then $\mathrm{ml}(G) \leq k$.*

Since a branch vertex has degree at least 3, it is obvious that $\mathrm{ml}(G) \geq \mathrm{s}(G)+2$, thus Theorem 3 is a generalization of Theorem 2 indeed. Theorem 3 was rediscovered in 2012 by Kano, Kyaw, Matsuda, Ozeki, Saito, and Yamashita [3] and they also proved a stronger version.

Theorem 4. (Kano et al., 2012) *Let G be a connected claw-free graph of order n and let k be a nonnegative integer. If $\delta_{k+1}(G) \geq n-k$, then G has a spanning 3-tree with at most k leaves.*

The main result of the paper is the following theorem.

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Theorem 5. *Every connected claw-free graph G has a DFS-tree T such that no two of the d-leaves of T have a common neighbour. Moreover, if v is not a cut vertex of G, then T can be chosen such that it is rooted at v.*

Though the proof of Theorem 5 is really short, it is omitted here due to the page limit. On the other hand, we sketch how Theorem 5 implies Theorem 4. Let G be a connected claw-free graph of order n with $\delta_{k+1}(G) \geq n-k$ and let T be a DFS-tree guaranteed by Theorem 5. The set of d-leaves D of T is obviously an independent set, thus the degree sum of the vertices of D is at most $n - |D|$, since all vertices in $V(G) - D$ has at most one neighbour in D. Hence $|D| \leq k$, that is T has at most $k+1$ leaves. Notice that T, like any DFS-tree of a claw-free graph is a 3-tree. In order to find a spanning 3-tree with at most k leaves, we need a further local improvement step, which is omitted here due to lack of space.

Theorem 5 has some other connections with results concerning claw-free graphs, of which we only mention the following corollary.

Corollary 6. *Let G be a connected claw-free graph of diameter at most 2 and let v be a non-cut vertex of G. Then there exists a hamiltonian path of G starting at v.*

Proof. By Theorem 5, there exists a DFS-tree T of G rooted at v, such that no two of the d-leaves of T have a common neighbour. Since the diameter of G is at most 2, this is possible only if T has just one d-leaf, which finishes the proof. $\square$

Corollary 6 is a stronger form of a result of Ainouche, Broersma, and Veldman [1] stating that every connected claw-free graph of diameter at most 2 is traceable (actually they also proved the more general theorem that all m-connected claw-free graphs G with $\alpha(G^2) \leq m+1$ are traceable).

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On the Recognition of Simple-Triangle Graphs and the Restricted 2-Chain Subgraph Cover (Extended Abstract)

Asahi Takaoka*

Abstract

A simple-triangle graph (also known as a PI graph) is the intersection graph of a family of triangles defined by a point on a horizontal line and an interval on another horizontal line. The recognition problem for simple-triangle graphs was a long-standing open problem, and recently a polynomial-time algorithm has been given [G. B. Mertzios, The Recognition of Simple-Triangle Graphs and of Linear-Interval Orders is Polynomial, SIAM J. Discrete Math., 29(3):1150–1185, 2015]. This paper shows a simpler recognition algorithm for simple-triangle graphs. To do this, we provide a polynomial-time algorithm to solve the following restricted 2-chain subgraph cover problem: Given a bipartite graph G and a set F of edges of G, find a 2-chain subgraph cover of G such that one of two chain subgraphs has no edges in F. This is an extended abstract, and the full version can be found on arXiv.

Keywords: Chain cover, Graph sandwich problem, PI graphs, Simple-triangle graphs, Threshold dimension 2 graphs

1 Introduction

Let L_1 and L_2 be two horizontal lines in the plane with L_1 above L_2. A *simple-triangle graph* is the intersection graph of a family of triangles spanned by a point on L_1 and an interval on L_2. That is, a simple undirected graph is called a simple-triangle graph if there is such a triangle for each vertex and two vertices are adjacent if and only if the corresponding triangles have a nonempty intersection. See Figures 1(a) and 1(b) for example. Simple-triangle graphs are also known as *PI graphs* [3, 5], where *PI* stands for *Point-Interval*. Simple-triangle graphs were introduced in [5] as a generalization of both interval graphs and permutation graphs, two well-known geometric intersection graphs.

Another generalization of both interval graphs and permutation graphs is trapezoid graphs, which have been introduced independently in [5, 6]. A *trapezoid graph* is the intersection graph of a family of trapezoids spanned by an interval on L_1 and another interval on L_2. Simple-triangle graphs are known as a proper subclass of trapezoid graphs. A separating example of these classes is the Berlin graph $\overline{B}$ in Figure 1(c). This graph is a trapezoid graph as shown in Figure 1(d), but it is not a simple-triangle graph, since the graph $\overline{B}$ is not alternately orientable [7, 8], while any simple-triangle graph is alternately orientable [5].

Recently, the graph isomorphism problem has shown to be isomorphism-complete for trapezoid graphs [15]. Since the graph isomorphism problem can be solved in linear time for interval graphs [10] and for permutation graphs [4], it has become an interesting question to give the structural characterization of graph classes lying strictly between permutation graphs and trapezoid graphs or between interval graphs and trapezoid graphs. As stated above, the class of simple-triangle graphs is one of such classes. Although a lot of research has been done for interval graphs, for permutation graphs, and for trapezoid graphs (see [8, 14] for example), there are few results for simple-triangle graphs [2, 3, 5]. It is only recently that a polynomial-time recognition algorithm have been given in [11, 12].

The recognition algorithm first reduces the recognition problem to the *linear-interval cover* problem. The algorithm then reduces the linear-interval cover problem to *gradually mixed* formulas, a tractable subclass of 3-satisfiability (3SAT). Finally, the algorithm solves the gradually mixed formula by reducing it to 2-satisfiability (2SAT), which can be solved in linear time (see [1] for example). The total running time of the algorithm is $O(n^2\bar{m})$, where n and $\bar{m}$ is the number of vertices and non-edges of the given graph, respectively.

In this paper, we introduce the *restricted 2-chain subgraph cover* problem as a generalization of the linear-interval cover problem. Then, we show that our problem is directly reducible to 2SAT. This result does not improve the running time, but it can simplify the previous algorithm for the recognition of simple-triangle graphs.

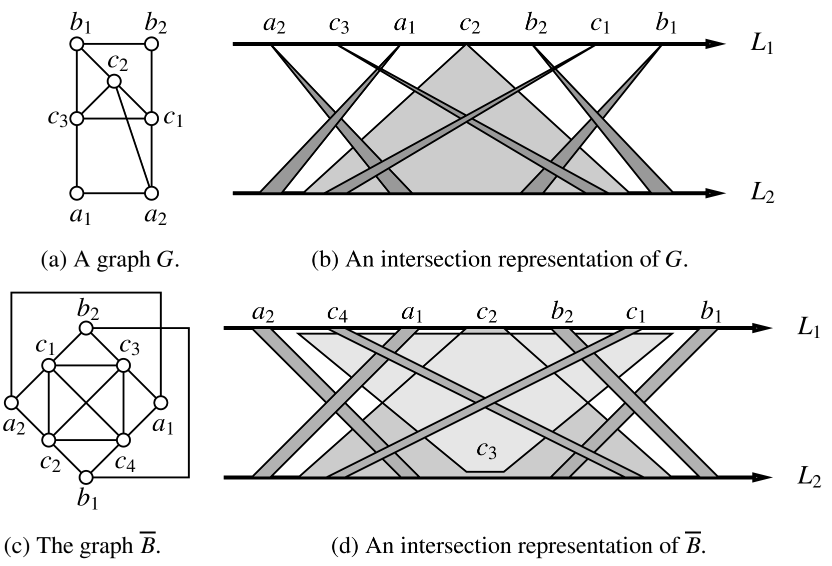
This paper is an extended abstract, and a full version can be found in [16].

2 Restricted 2-chain subgraph cover

Let $2K_2$ denote a graph consisting of four vertices u_1, u_2, v_1, v_2 with two edges u_1v_1, u_2v_2. A bipartite graph $G = (U, V, E)$ is called a *chain graph* [17] if it has no $2K_2$ as an induced subgraph. Equivalently, a bipartite graph G is a chain graph if and only if there is a linear ordering $u_1, u_2, \ldots, u_n$ on U such that $N_G(u_1) \subseteq N_G(u_2) \subseteq \ldots \subseteq N_G(u_n)$, where $N_G(u)$ is the set of vertices adjacent to u in G. A *chain subgraph* of G is a subgraph of G that has no induced $2K_2$. A bipartite graph $G = (U, V, E)$ is said to be *covered* by two chain subgraphs $G_1 = (U, V, E_1)$ and $G_2 = (U, V, E_1)$ if $E = E_1 \cup E_2$, and the pair of chain subgraphs (G_1, G_2) is called a *2-chain subgraph cover* of G.

Restricted 2-Chain Subgraph Cover	
Instance:	A bipartite graph $G = (U, V, E)$ and a set F of edges of G.
Question:	Find a 2-chain subgraph cover (G_1, G_2) of G such that G_1 has no edges in F.

*Department of Information Systems Creation, Kanagawa University, Rokkakubashi 3-27-1 Kanagawa-ku, Kanagawa, 221–8686, Japan E-mail: takaoka@jindai.jp



(a) A graph G. (b) An intersection representation of G.

(c) The graph $\overline{B}$. (d) An intersection representation of $\overline{B}$.

Figure 1: A simple-triangle graph G and the Berlin graph $\overline{B}$ with their intersection representations.

Notice that G_2 has all the edges in F. Let $\hat{E}$ be the set of non-edges of G such that $uv \in \hat{E}$ if and only if $uv \notin E$ for every $u \in U$ and $v \in V$. Let $m = |E|$, $\hat{m} = |\hat{E}|$, and $f = |F|$. The following is our main result. We note that the algorithm is based on the techniques used in [9, 13].

Theorem 1. *The restricted 2-chain subgraph cover problem can be solved in $O(m\hat{m} + \min\{m^2, \hat{m}(\hat{m} + f)\})$ time.* □

3 Concluding remarks

We finally note that for simple-triangle graphs, forbidden structure characterizations as well as the complexity of the graph isomorphism problem still remain interesting open questions.

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A Note on Distance Domination in Maximal Outerplanar Graphs (Extended Abstract)

Liang Zhao, GSAIS, Kyoto University, Japan

Let $G = (V, E)$ be an undirected graph with a set V of n nodes and a set E of m edges. A node v is said to *distance-k dominate* a node w if w is reachable from v by a path consisting of at most k edges. A set $D \subseteq V$ is said a *distance-k dominating set* if every node $w \in V$ is distance-k dominated by some node $v \in D$. The size of a minimum distance-k dominating set, denoted by $\gamma_k(G)$, is called the *distance-k domination number* of G. In particular, $\gamma_1(G)$ is the well-known *domination number*.

Domination is one of the fundamental topics in graph theory, see [1, 4, 5, 6, 7] for some recent works. This paper considers the distance-k domination numbers for a *maximal outerplanar graph* (MOG). A graph is said *outerplanar* if it can be drawn in the plane without crossing and the nodes belong to the unbounded outer face. It is *maximal* if adding any extra edge breaks this property. It is known that a graph is outerplanar if and only if it does not contain K_4 or $K_{2,3}$ as a minor ([2]), and a MOG is a *visibility graph*, i.e., a *triangulation* graph, of a simple polygon with n vertices ([3]).

In general, it is not trivial to determine $\gamma_k(G)$ even for a MOG. Since the outer boundary C of a MOG is a Hamilton cycle in G, we see $\gamma_k(G) \leq \gamma_k(C) = \left\lceil \frac{n}{2k+1} \right\rceil$. In particular, $\gamma_k(G) = 1$ if $n \leq 2k$. Thus in the following we only consider for $n \geq 2k+1$.

There is a conjecture that $\gamma_k(G) \leq \left\lfloor \frac{n}{2k+1} \right\rfloor$ and it is tight. For $k = 1, 2$, it is true ([5, 1]). In this paper, we give a simpler proof and further prove for $k = 3$, all by a linear-time construction algorithm. In fact, we show a stronger result that for all $r = n \bmod (2k+1) \leq 6$, there exist at least $2k+1-r$ *distinct* distance-k dominating sets of size at most $\left\lfloor \frac{n}{2k+1} \right\rfloor$, and they can be found in linear time.

Let $P = u - w - v$ denote a path with nodes u, w, v and edges (u, w), (w, v). A *triangle ear* (simply *ear* in the following) with respect to a graph $G = (V, E)$ is such a path $P = u - w - v$ that $w \notin V$, $u, v \in V$, and $(u, v) \in E$ (see an illustration in Figure 1). We use $G + P$ to denote the graph obtained by adding P to G, and similarly $G + P_1 + \cdots + P_i = (G + P_1 + \cdots + P_{i-1}) + P_i$, $i = 2, 3, \ldots$.

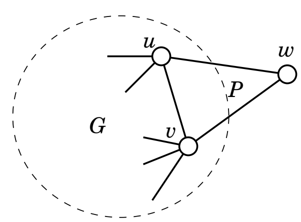

Figure 1: An illustration of a triangle ear $P = u - w - v$ with respect to G.

Theorem 1 *For any integers $k \geq 1$, $p \geq 1$, $0 \leq r \leq \min\{6, 2k\}$ and $n = p(2k+1) + r$, $\gamma_k(G) \leq p = \left\lfloor \frac{n}{2k+1} \right\rfloor$ for any graph $G = C + P_1 + \cdots + P_r$, where C is a simple cycle of $p(2k+1) = n - r$ nodes, P_i are triangle ears with respect to $C + P_1 + \cdots + P_{i-1}$. Moreover, at least $2k+1-r$ distinct distance-k dominating set of G consisting of at most p nodes of C can be found in $O(n)$ time.* □

Corollary 1 $\gamma_k(G) \leq \left\lfloor \frac{n}{2k+1} \right\rfloor$ *for a MOG G of $n \geq 2k+1$ nodes and $k = 1, 2, 3$.* □

Corollary 2 *For a MOG with $n \geq 2k+1$ nodes, at least $2k+1-r$ distinct distance-k dominating set of size at most $\left\lfloor \frac{n}{2k+1} \right\rfloor$ can be found in $O(n)$ time if $k \leq 3$, where $r = n \bmod (2k+1) \leq 2k$.* □

Remark We remark that Theorem 1 can be applied to non-MOGs as well, as far as there exists an ear decomposition. Also the bound $\left\lfloor \frac{n}{2k+1} \right\rfloor$ is tight (see the tight example in [5]).

The proofs are omitted in this extended abstract. We just mention that the proof for the main theorem (Theorem 1) consists of many case studies and is difficult to be generated. Now we are working on a simpler and more general proof to study the case of $r \geq 7$ (or to develop a counterexample). As another future work, we are considering to improve the results for guarding numbers and vertex cover numbers as considered by [1] for $k = 2$.

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A Fast Hybrid Parallel Method for the Traveling Salesman Problem

Nagatsuji Ryouta*[1,a)] Ueda Youta[1] Ebara Hiroyuki[2,b)]

1. Introduction

In the real world, we need to solve problems quickly, such as delivering packages by following a single route, holing to a printed board by using machine tools. When solving optimization problems by computer, it requires us to define an objective function and constraint conditions. For example, since package delivery and holing to an electronic substrate require us to visit all vertices in a network only once, we can express these problems as typical combinatorial optimization problems known as the Traveling Salesman Problem (TSP). Combinatorial optimization problems such as the TSP can be solved approximately by some meta heuristics of Swarm Intelligence (SI) such as the Genetic Algorithm (GA)[1], Ant Colony System (ACS)[2], and Consultant Guided Search (CGS)[3]. These algorithms can be enhanced with Particle Swarm Optimization (PSO) parameter tuning[4],[5]. In this study, we propose a parallelization method to power up the tuned SI to solve the TSP quickly. We aim to develop a fast algorithm, rather than an algorithm that provides an accurate solution.

2. Traveling Salesman Problem(TSP)

Given some cities as the vertices, we are required to visit all cities only once as the solution of the tour, which should minimize the total cost of returning to the start city. Increasing the number of cities makes it more difficult to achieve the goal at an accelerated pace. The TSP had been studied as well. Researchers have reported the power of meta heuristics[1],[2],[3],[4],[5] to solve the problem. In this study, we use the instances of TSP released by TSPLIB[6].

$$\min_x f(x) = \min_x \sum_{i \in [0, n-1]} C(x_i, x_{(i+1) mod n}) \tag{1}$$

$$x \in \{x | i, j \in [0, n-1] \land i \neq j \land x_i \neq x_j \land x_i, x_j \in V, \} \tag{2}$$

where the function $C(i, j)$ represents the cost of traveling from city i to j.

3. Consultant Guided Search(CGS)

CGS[3] algorithm models the advice a person receives from a consultant to decide the action to take. This algorithm has some individuals known as a Virtual Person that creates the tour as strategy by visiting the next city from the current city such as the ACO individual who creates the tour. CGS provides good performance for graph problems. The Virtual Person has two modes, i.e., Sabbatical and Normal; however, the approach that needs to be taken to build a strategy is difficult. In terms of formulas, (3) represents Sabbatical mode strategy, whereas (4) represents Normal mode strategy.

$$v_{next} = \begin{cases} \arg\min_{i \in V_R} C(v_{current}, i) & (rand \leq a_0) \\ \arg\operatorname{prob}_{i \in V_R} C(v_{current}, i)^{-\beta} & otherwise \end{cases} \tag{3}$$

$$v_{next} = \begin{cases} advice(c, v_{current}, V_R) & (\exists c \land rand \leq q_0) \\ \arg\min_{i \in V_R} C(v_{current}, i) & (rand \leq b_0) \\ \arg\operatorname{prob}_{i \in V_R} C(v_{current}, i)^{-\beta} & otherwise \end{cases} \tag{4}$$

$$c = \arg\operatorname{prob}_{i \in AllNormalModePersons} Rep(i)^{\alpha} Pre(i)^{\gamma} \tag{5}$$

where the variable c is the consultant selected by the building strategy of the Normal mode Virtual Person.

4. Proposed method

The team lead by Mostafa Mahi conducted research on the hybrid method Ant Colony Optimization (ACO), PSO, and 3opt[4]. All ants as ACO individuals search for a solution, and then evaluate and update all ant parameters by relating to PSO particle location. Finally, they applied a 3opt local search to find the best solution only once. Computational experiments show their solution to be more accurate than that of other ACO hybrid methods.

We previously developed CGS-PSO[5], which shows powerful performance. CGS-PSO relates to six parameters of the Virtual Person and PSO particle position to tune the Virtual Person parameters. Especially, the results show very good performance for 5,000 cities class TSP instances.

In this study we aim to demonstrate an enhanced ability in terms of SI tuning; thus, we propose the parallelization method. First, we apply a 2opt local search to all strategies held by the CGS-PSO Virtual Person and tune the parameters of the Virtual Person with PSO. We name this sub-system CGS-PSO-2opt. Second, we prepare many CGS-PSO-2opt sub-systems in parallel to enable it to search for the solution in parallel. Each CGS-PSO-2opt sub-system transports PSO particle information to each other and receives its information from the other sub-systems for embedding the PSO strategy such as in equation (6). We name this parallel system Q-CGS-PSO-2opt. Finally, in all CGS-PSO-2opt sub-systems in the final execution of Q-CGS-PSO-2opt, we select the best solution provided by this sub-system, to which we then apply a 3opt local search.

$$\Delta^2 x = (x_{i,best} - x) r_1 c_1 + (x_{best} - x) r_2 c_2 + (Q - x) r_3 c_3, \tag{6}$$

where the variable Q is the parameter particle sent from other computing nodes. The time at which Q information is sent, is when the CGS-PSO-2opt sub-system finds a more suitable solution, and it is the parameter of the best Virtual Person in the iteration. Fig. 1 shows the algorithm flow chart of the proposed system. This allows PSO to quickly find an optimal area of parameter space, and allows CGS to quickly find a accurate TSP solution. In addition, to improve the efficiency, we apply a 2opt local search to the Virtual Person strategy and 3opt local search to

[1] Graduate School of Science and Engineering-Systems Science Department-Electrical and Electronic Information Engineering, Kansai University, Algorithm Engineering Lab, 564-8680, Japan
[2] Kansai University, Faculty of Engineering Science, Department of Electrical and Electronic Engineering, 564-8680, Japan
* the presenter
a) ryoutafunphy@gmail.com
b) ebara@kansai-u.ac.jp

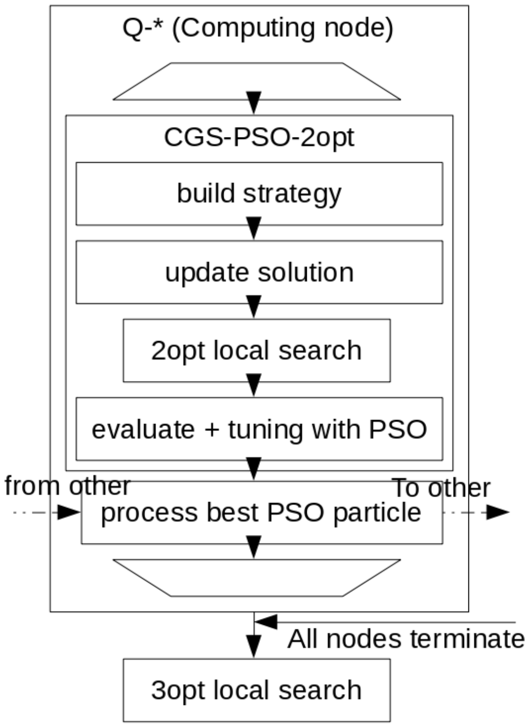

Fig. 1 flow chart of Q-CGS-PSO-2opt-3opt

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the best solution after Q-CGS-PSO-2opt ended. We name this method Q-CGS-PSO-2opt-3opt. We compare it with GA-EAX.

5. Computational experiments

We compare the ability of Q-CGS-PSO-2opt-3opt and GA-EAX by installing computing nodes into the one computing server. Computing nodes execute Q-CGS-PSO-2opt-3opt as the Message Passing Interface Application. In addition, GA-EAX is executed with one thread of the same machine to enable this method to be compared with Q-CGS-PSO-2opt-3opt. Table 1 provides the experimental environment. Table 2 shows the resource usage for Q-CGS-PSO-2opt-3opt, and the CGS-PSO parameters have been reported[5]. Table 3 contains the GA-EAX parameters.

We perform 10 iterations for each TSP instance. Table 4 lists the time taken before the error rates reach the specified target. The column containing the average[s] presents the average time before the error rate reaches the target. The fastest[s] presents the time for the minimum error rate to reach the target. The average represents the average total error rate to reach the target [s]. In addition, min presents the minimum error rate to reach the target in total[s]. The results show the proposed method of Q-CGS-PSO-2opt-3opt is much faster than GA-EAX. Because the final solution of Q-CGS-PSO-2opt-3opt depends on the 3opt local search method, the average and min error rate value does not improve. As the results show, we consider the proposed method of Q-CGS-PSO-2opt-3opt to be useful for solving the TSP, because it is very fast.

In the fl3795 instance, the error rate of GA-EAX is not acceptable, because, it does not permit the parameters to be sufficiently tuned. We insist that Q-CGS-PSO-2opt-3opt never requires parameter tuning.

6. Conclusion

We proposed a parallelization method for the PSO tuned SI with the aim of solving combinatorial optimization, such as the TSP, in an efficient manner. We evaluated our proposed method by comparing Q-CGS-PSO-2opt-3opt with GA-EAX for TSP. The computational experimental results show the error convergence speed to be much faster than that of GA-EAX. Since we could not find optimal parameters for GA-EAX, the results it achieves are not optimal. Although the goodness of the Q-CGS-PSO-2opt-3opt solution depends on the 3opt local search, we succeeded in increasing the speed of 3opt running with Q-CGS-PSO-2opt-3opt. We consider it to have high parallelization potentiality. In future, we plan to solve larger TSP instances by preparing additional parallel computing resources.

Table 1 computational experiment environment

name	value
OS	CentOS 6.5
cpu	Intel-Xeon E5-2620v2 x2
clock[GHz]	2.1
threads	24
chipset	C600
memory	DDR3-1600 ECC registered 4GBx8 : 32GB
MPI	Intel MPI Library for Linux, 5.0 Update 2
compiler	icpc (ICC) 15.0.1 20141023

Table 2 Q-CGS-PSO-2opt-3opt resource usages

name	value
n.o. computing nodes	20(Q-CGS-PSO-2opt) + 1(3opt)
n.o. Virtual Person	10 / computing node

Table 3 GA-EAX parameters

name	value
n.o. individuals	120
selection	roulette
xover rate	0.8
mutation rate	0.005
copy rate	0.005
off spring	10

Table 4 results: reach time reach target error rate

algorithm	instance	total[s]	target	average[s]	fastest[s]	average	min
Q-CGS-PSO-2opt-3opt	d493	18	2.5%	**14**	**10**	2.0476%	1.3771%
GA-EAX	d493	4,000	2.5%	376	300	**1.2345%**	**0.8514%**
Q-CGS-PSO-2opt-3opt	pr1002	138	3.5%	**99**	**50**	3.1914%	2.4579%
GA-EAX	pr1002	6,000	3.5%	4,087	2,050	**2.1407%**	**1.6916%**
Q-CGS-PSO-2opt-3opt	d2103	1,643	3.0%	**385**	**24**	**2.0142%**	**1.2070%**
GA-EAX	d2103	72,000	3.0%	37,700	14,500	2.5954%	1.4829%
Q-CGS-PSO-2opt-3opt	fl3795	13,861	4.0%	**1,728**	**60**	**2.3426%**	**1.6996%**
GA-EAX	fl3795	432,000	7.0%	225,200	61,000	6.9522%	5.2273%

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Routing Problems with Last-Stretch Delivery

Mohd Shahrizan bin Othman[1,2] Aleksandar Shurbevski[1]
Yoshiyuki Karuno[3] Hiroshi Nagamochi[1]

[1]Department of Applied Mathematics and Physics, Kyoto University
[2]Technical University of Malaysia Malacca
{shah, shurbevski, nag}@amp.i.kyoto-u.ac.jp
[3]Faculty of Mechanical and System Engineering,
Kyoto Institute of Technology
karuno@kit.ac.jp

1 Introduction

Unmanned Aerial Vehicles (UAV-s), or more popularly known as *drones*, have enjoyed a widespread adoption for many purposes in recent years. One of the areas which gathers a lot of attention with announcements for using drones is parcel delivery [1, 2]. One interesting proposal is to use drones for *last-stretch*, or *last-mile delivery*, where a drone is used in tandem with a delivery truck to perform deliveries to end customers. Murray and Chu [2] have proposed the Flying Sidekick Traveling Salesman Problem (FSTSP), where they model a problem where a drone is dispatched from a moving truck to deliver a parcel to an end customer while the truck continues on its route, and the drone rejoins the truck after completing the delivery.

In this study we examine a somewhat restricted problem model, where the route of a delivery truck is already given, and a drone of unit capacity is dispatched for the last-stretch delivery while the truck continues on its route. Moreover, the drone can rendezvous with the truck only at certain defined locations along the truck's route. Then, given a set of customers to be served and a set of rendezvous points where the drone can meet with the truck to pick up a parcel, we ask what is the quickest way of delivering all parcels to respective end customers.

We examine two particular cases of the problem arising in the scenario outlined above. In the first setting, the drone immediately takes off from the truck after getting a parcel, while in the second, the drone may "hitch a ride" on the truck before proceeding to its next delivery. We term the former the Alternating Last-Stretch Delivery Problem, or ALSDP for short, whereas the later the Last-Stretch Delivery Problem, or LSDP.

With this work, we propose graph problem models for the ALSDP and LSDP. Next, we show that the graph problems are NP-hard even in metric graphs. Furthermore, we identify a special instance type of the ALSDP and the LSDP that can be solved optimally in polynomial time. Finally, we propose a polynomial time approximation algorithm, and show that the algorithm has a factor 2-approximation guarantee in metric graphs.

2 Problem Models

Let C be the set of customers to which parcels need to be delivered, and let R be the set of points at which the delivery drone can rendezvous with the truck along the truck's predetermined route. The order in which the truck passes points in the set R along the path defines a total order $\prec$ on R. Let $m = |R|$ and $n = |C|$. We assume that $m > n$ always holds. Because the drone has unit payload capacity, it never visits two customers in C consecutively, and between two points in R, the drone effectively "hitches a ride" on the truck. With this observation, we introduce the following distance functions:

- $t(u, v)$ - the time it takes for the truck to move from point u to point v along its predetermined route, $u, v \in R$,
- $d(u, v)$ - the time it takes for the drone to travel between rendezvous point $u \in R$ to customer $v \in C$.

We assume that for all $u, v \in R$ and any $q \in C$, it holds that

$$t(u, v) \le d(u, q) + d(q, v), \tag{1}$$

which is a natural assumption that the drone cannot take a shortcut between rendezvous points by visiting a customer. Furthermore, by the assumption that points in R appear along the truck's route, we assume that for any $u, v, q \in R$, $u \prec v \prec q$, it holds

$$t(u, q) = t(u, v) + t(v, q). \tag{2}$$

In the scenario where the drone is required to take off immediately carrying a parcel and only rendezvouses with the truck to pick up the next parcel to be delivered, we model the problem by a bipartite graph $G = (R \cup C, E)$, and this graph is weighted by an edge weight function $w(u, v) = d(u, v)$, $u \in R$, $v \in C$, $\{u, v\} \in E$. Thus, we get the following problem model.

The Alternating Last-Stretch Delivery Problem - ALSDP

Input: A bipartite graph $G = (R \cup C, E)$, a weight function $w : E \to \mathbb{R}_+$, and a total order $\prec$ on R, where $s \in R$ is the unique minimum element of R with respect to $\prec$, and $t \in R$ is the maximum.

Output: An s,t-path P in G, such that
(i) all vertices in C are visited in P,
(ii) vertices in R are visited in the total order $\prec$,
(iii) vertices in R and C appear alternately in P, and
(iv) $w(P)$ is minimized.

We say that a path P that satisfies conditions (i) to (iii) above is *feasible* to the ALSDP. In particular, we say that a path satisfying condition (iii) is *R,C-alternating.*

Under the assumption that the drone may "hitch a ride" on the truck between consecutive deliveries, we express a relaxed version of the ALSDP. Now, instead of a bipartite graph, we consider a graph $G = (R \cup C, E)$ such that $E = R \times C \cup R \times R$, and C is an independent set. We define a weight function $w : E \to \mathbb{R}_+$ in this graph to be $w(u,v) = d(u,v)$ if $u \in R$ and $v \in C$, and $w(u,v) = t(u,v)$ otherwise. Then, the problem model becomes as follows.

The Last-Stretch Delivery Problem - LSDP

Input: A graph $G = (R \cup C, E = R \times C \cup R \times R)$ where C is an independent set; a weight function $w : E \to \mathbb{R}_+$; and a total order $\prec$ on R, where $s \in R$ is the unique minimum element of R with respect to $\prec$, and $t \in R$ is the maximum.

Output: An s,t-path P in G, such that
(i) all vertices in C are visited in P,
(ii) vertices in R are visited in the total order $\prec$, and
(iii) $w(P)$ is minimized.

We say that a path P that satisfies conditions (i) and (ii) above is *feasible* to the LSDP.

3 NP-hardness

Theorem 1 *The recognition version of the ALSDP is NP-hard.*

Theorem 2 *The LSDP is NP-hard even when all edge weights are restricted to be 1 or 2.*

Both Theorem 1 and Theorem 2 can be shown by a reduction from the Hamiltonian Path Problem.

4 Polynomially Solvable Case

Note that when $|R| = |C|+1$, the LSDP is equivalent to the ALSDP, in the sense that in both problems a feasible path is R,C-alternating, and we write (A)LSDP. An easy observation gives us a useful insight for constructing a polynomial time algorithm via minimum cost bipartite matching. Let vertices $u_1, u_2, \ldots, u_m \in R$ be numbered according to $\prec$.

Observation 1 *In any feasible path P, if a vertex $v \in C$ is visited immediately after $u_i \in R$, then $u_{i+1} \in R$ is visited immediately after v.*

Theorem 3 *An instance $(G = (R \cup C, E), w, \prec)$ of the (A)LSDP with $|R| = |C| + 1$, can be solved in polynomial time.*

5 Approximation Algorithm

We say that the ALSDP and the LSDP are given in a metric setting if the model's graph's edges are weighted by a metric function w that satisfies Eq. (2) with respect to the set R and the total order $\prec$.

Observation 2 *Let $(G = (R \cup C, E), w, \prec)$ be an instance of the LSDP such that the edge weight function w is metric. Then, there exists an optimal path P in G, such that vertices in R and C appear alternately.*

By Observation 2, if we find an optimal solution to the ALSDP, it will also be an optimal solution for the LSDP.

Theorem 4 *In a metric setting, both the ALSDP and the LSDP can be approximated within a constant factor 2.*

Theorem 4 can be seen with a few observations on the structure of a feasible R,C-alternating path P. Let vertices $u_1, u_2, \ldots, u_m \in R$ be numbered according to $\prec$. Notice that in a metric setting, due to Eq. (2), it holds

$$\sum_{i=1}^{m-1} w(u_i, u_{i+1}) \le w(P).$$

In addition, the edges in P that are incident to vertices in C can be partitioned into two disjoint matchings, M_1 and M_2, and we have that

$$w(M_1) + w(M_2) \le w(P).$$

The above observations give us a simple heuristic for constructing an R,C-alternating u_1, u_m-path.

6 Future Work

As future work, it remains to investigate whether an approximation algorithm with an approximation ratio better than 2 exists or not. Further, it would be interesting to analyze some extensions of both routing problems, the LSDP and the ALSDP, possibly in metric settings but with edge weight bias to account for additional transportation effort exerted by the drone when delivering a parcel [3], as well as to examine a combinatorial optimization based model for a routing problem including both the delivery truck and the drone [2].

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Efficient Enumeration of Induced Matchings in Graphs without Short Cycles

Kazuhiro Kurita[1], Kunihiro Wasa[2], Takeaki Uno[2], and Hiroki Arimura[1]

[1] IST, Hokkaido University, Sapporo, Japan, {k-kurita,arim}@ist.hokudai.ac.jp
[2] National Institute of Informatics, Tokyo, Japan, {wasa,uno}@nii.ac.jp

Backgrounds In this paper, we study efficient algorithms for enumerating all induced matchings in an input graph. An *induced matching* in a graph G, also called a *risk-free marriage*, is a natural extension of matchings, and defined as a set M of edges such that the vertex set of M induces M itself. Induced matchings are an important extension of matchings, and they have potential applications to inexact image search or ontology matching [8].

In contrast to the case for matchings [5, 9], the maximum induced matching problem is computationally hard. Cameron [3], and Stockmeyer and Vazirani [10] showed that it is NP-hard to find a maximum induced matching in a graph. It is still NP-hard for bipartite, line, and planar graphs [2–4, 8], while it is polynomial time solvable for the following classes: interval, chordal, weakly chordal, circular-arc, trapezoid, and co-comparability graphs [2,4,8]. However, to our best knowledge, enumeration problems for induced matchings have not been studied well.

Recently, a number of efficient enumeration algorithms for all substructures of a given graph [12] have been developed for various classes such as *spanning trees*, *bounded-sized trees*, *cycles and st-paths*, *chordless cycles*, and *induced trees* [1, 6, 7,13,14]. Most of these algorithms output solutions in polynomial amortized time or delay in input size. Among these results, however, there are a few algorithms with optimal complexity in amortized sense, namely *constant amortized time* per output [11].

Goal of this research In this paper, we present an efficient algorithm for enumerating induced matchings in a graph. Particularly, we designed the algorithm so that it runs efficiently for graphs without short cycles. We show that the algorithm enumerates all induced matchings in constant amortized time per output for graphs with girth 5 or more, i.e., without no cycles of length 4 or less. Furthermore, we also show that the maximum induced matching problem is NP-complete for this class in the same manner as in [3, 10].

There are few algorithms for enumerating all matchings and induced matchings. Recently, it is shown by Uno [11] that all matchings in a graph can be enumerated in constant amortized time. He used an amortization technique called *Push Out* to amortize the cost of each iteration to many descendants. Starting from a naive $O(m)$ time enumeration algorithm based on binary partition, we devise an improved algorithm with $O(\Delta^2)$ amortized complexity, and finally, a constant amortized time algorithm by similar ideas to [11] of grouping solutions with a similar structure.

Related work and open problems The maximum matching problem has been extensively studied for long years [5,9]. Edmonds [5] presented an $O(mn^2)$ time algorithm, and Hopcroft and Karp [9] improved its complexity to $O(m\sqrt{n})$ time for bipartite graphs.

Cameron [4] pointed out an interesting one-to-one correspondence between induced matchings in a graph and independent sets in the square $L(G)^2$ of the line-graph of G. Thus, constant amortized time enumeration algorithms for the former class may give some insight to enumeration of independent sets in the latter classes, which is still open.

In this paper, we also discuss an enumeration problem for *k-distance matchings* in a graph. A k-distance matching M of a graph G is an edge set such that for any pair of edges e and f in M, the distance between the nearest pair of end points of e and f is at least k. Note that a k-distance matching is a generalization of matchings and induced matchings. However, it seems to be difficult to develop an efficient enumeration algorithm for k-distance matchings by using the approach of us and Uno [11] when $k \geq 3$.

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Optimal Play of Piglet with Three Players

François Bonnet[1], Todd W. Neller[2], and Simon Viennot[1]

[1]Graduate School of Advanced Science and Technology, JAIST, Japan
[2]Dept. of Computer Science, Gettysburg College, Pennsylvania, USA

1 Game description

Piglet is a simple coin game played between two or more players. "*The object of Piglet is to be the first player to reach 10 points. Each turn, a player repeatedly flips a coin until either a tail is flipped or else the player holds and scores the number of consecutive heads flipped.*" [3]

Piglet is Neller and Presser's coin simplification of the dice game Pig, the simplest known folk ancestor of the jeopardy dice game family. The commercial games Pass the Pigs (a.k.a. Pigmania) and Farkle are perhaps the best known games of this family.

2 Motivation

Neller and Presser proposed a Value-Iteration-based method to compute the optimal strategy of the 2-player version of Piglet and Pig [3], and of many variants of the game [4]. Until now, no analysis with more players has been done.

In this research, we investigate the 3-player version of Piglet. While we expected it to be a trivial generalization of the 2-player version, it appears to be quite different. Indeed 3-player games are generally much harder to analyze. As described for example in [1, Chapter 43] for a Poker-like game, coalitions between players are possible in 3-player games, so it is difficult to even define "optimal play".

3 Contributions

3.1 Playing Independently

Instead of extending the Value Iteration method used in [3], we design a new computation method for Nash equilibria (NE) [2] by exhaustively listing the payoffs of each possible pure strategy for each player.

This allows us to compute the NE of 3-player Piglet for the first time. Figure 1 shows the expected gain of each player when a win is rewarded $+2$ and a loss -1, for goal scores from 1 to 30. Without any surprise, expected gains decrease according to player order. Note that the gains are not monotonic, e.g. the expected gain of Player 2 decreases when the goal score goes from 3 to 4.

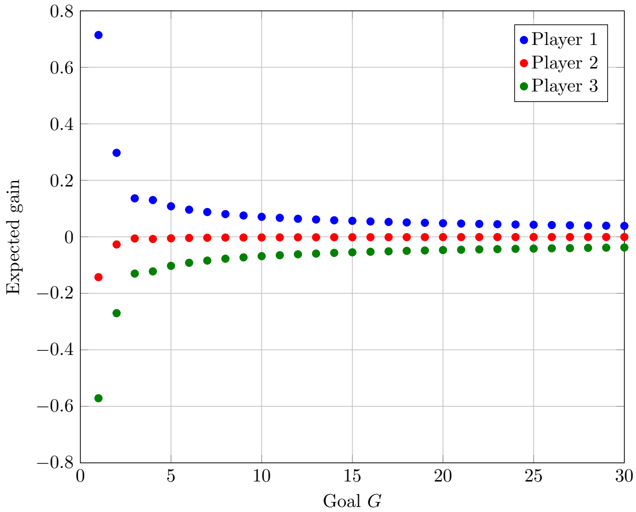

Figure 1: Expected gain versus goal score for three players using NE strategy.

3.2 Playing with Coalitions

Our computation method allows us to consider different objectives for the players. Assuming that each player seeks to maximize their winning probability, we obtain the results of Section 3.1, yet if we instead assume that two players seek to minimize the winning probability of the third, our computations show that such two-player coalitions are possible and effective in the game of Piglet.

Figure 2 shows the expected gain of Player 1 with and without a coalition of Players 2 and 3 for different goal scores. Though the difference is small, a coalition between Players 2 and 3 is an effective possibility above a goal score of 4 points. When Players 2 and 3 collude against Player 1, the expected gain of Player 1 decreases. It is worth mentioning that the expected gain of Player 3 also decreases in this coalition, representing self-sacrifice for the coalition's goal.

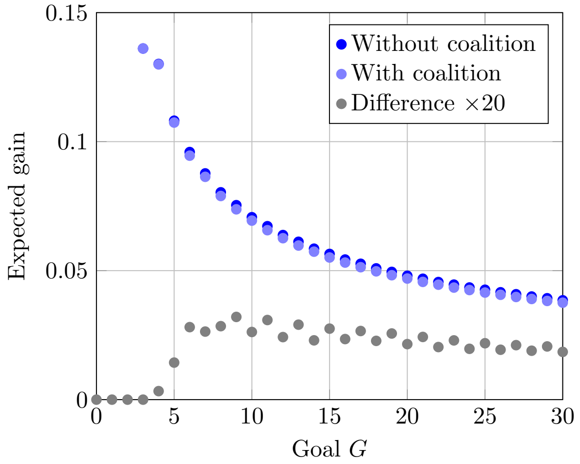

Figure 2: Expected gain of P1 with/without coalition of P2 and P3.

This possibility of coalitions is surprising because players have no direct interaction and thus no direct way to help each other in a game of Piglet. Still, coalitions are possible because one player can change his strategy when he observes that their partner is close to win. For example, if the two players in the coalition are both leading, the second-leading player can self-sacrifice by playing less aggressively. This reduces the pressure on the leading player, who can achieve a win more surely, to the detriment of the non-coalition player. This can be viewed as a probabilistic generalization of the "Kingmaker scenario".

3.3 Playing with a biased coin

We also computed what happens when playing Piglet with a biased coin. Figure 3 shows the expected gain of Player 1 as a function of the probability p of flipping a Head. Each curve corresponds to a given goal score G.

Figure 3 shows that when the probability of flipping a Head increases, the first player has globally a higher probability of reaching the goal score first. This is not surprising, since a greater chance of flipping a Head is likely to help the player increase its score faster. However, it is counter-intuitive that this expected gain is not monotonic with respect to the Head probability. For example, for a goal score $G = 10$ (red curve), Player 1's expected gain is higher when $p = 0.45$ than when $p = 0.5$.

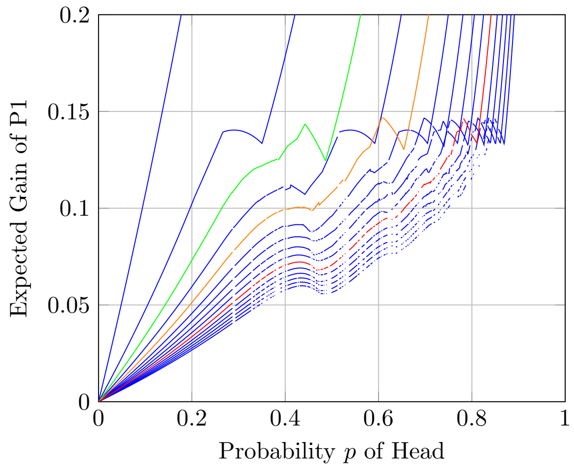

Figure 3: Each curve corresponds to a fixed goal G. Top curve is for $G = 1$, green curve is for $G = 3$, orange curve is for $G = 5$, red curve is for $G = 10$.

Also in Figure 3, some curves for different goal scores cross at some points. As noted in Section 3.1, this shows that the expected gain of Player 1 does not always monotonically decrease as the goal increases.

Some parts of the curves are missing in Figure 3. The reason is that for these instances of the game, there are multiple Nash equilibria with different payoffs, and thus no single optimal solution. Which solution concept to prefer is under consideration.

4 Conclusion

While Piglet is one of the simplest jeopardy games, optimal 3-player strategy is quite complex. We computed NEs for many instances of 3-player Piglet and observed that coalitions between players are possible. Also, in some cases, there are multiple NE with different payoffs, so that it is not obvious what should be considered "optimal play". There are still many open questions about 3-player Piglet, and even more about 3-player Pig.

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3-player NIM with preference

Koki Suetsugu

Graduate School of Human and Environmental Studies, Kyoto University
suetsugu.koki.72r@st.kyoto-u.ac.jp

1 Introduction

Almost all results of combinatorial game theory are based on two player games but some studies are done on multiplayer combinatorial games; Li [2], Straffin [3]. 3-player games are different from 2-player games in that the winner is not determined from the game position. For example, let's think about 3-player NIM. When there are two heaps and their sizes are one and two, the current player can take out all stones of one of the heaps, or take out one stone from the two-stone heap. This means that although she has no winning strategy, she can decide whether her next player or her previous player is the winner. Therefore, we should add more rules to determine a unique winner to each game situation like original combinatorial games.

In order that a unique winner is determined, Li introduced the notion of a Rank in n-player NIM [2]. He defined that the player who moves last is the winner. In addition, the k-th previous player of the winner is ranked as the $(k+1)$-th winner. He showed that if every player moves perfectly to obtain the best rank, the last player wins if and only if the mod-n sum of the number of stones of each heap in binary notation without carry is 0.

In this paper, we study 3-player games from another view.

2 Preference

In this article, we define the preference of a player to be a total ordering of the other players. Every player has her own preference, and other players know the content of each other's preference. Moreover, we also assume that it is common knowledge that every player knows the preferences of other players [1]. That is, each player knows that other players know the preferences of other players and so on. We suppose that if a player can't win, she behaves so that her most favorite player will win. If this is also impossible, then she behaves so that her second favorite player will win, ... and so on. With the assumption that all the players play in this way, and moreover that this fact is a common knowledge, the winner of this game is uniquely determined.

Let's think about 3-player NIM from this point of view. The players are A, B, and C, and they play in this cyclic order. There are 4 types of preferences: normal form, reverse form, semi-normal form, semi-reverse form.

Normal form: Player A likes player B more than player C (written $A : B > C$), $B : C > A$, and $C : A > B$.

Reverse form: Every player likes her previous player more than her next player, that is, $A : C > B, B : A > C, C : B > A$.

Semi-normal form: Two players like the same player and the popular player likes her next player more than her previous player. There are three possibilities,

1. $A : B > C, B : A > C, C : A > B$
2. $A : B > C, B : C > A, C : B > A$
3. $A : C > B, B : C > A, C : A > B$

but these three preference are essentially the same.

Semi-reverse form: Two players like the same player and the popular player likes her previous player more than her next player.

1. $A : C > B, B : A > C, C : A > B$
2. $A : B > C, B : A > C, C : B > A$
3. $A : C > B, B : C > A, C : B > A$

These three preference are essentially the same.

One can see that normal form games and games with Li's rank structure are essentially the same and we have the same result.

In semi-normal form or semi-reverse form, the popular player is quite advantageous. However, sometimes other players may win. The overall result for the semi-normal form is listed in Table 1 and for the semi-reverse form is listed in Table 2. Here, $(\alpha;\beta;\gamma)$ means that α,β,γ wins if the first player is A, B, C, respectively. And A is the popular player. We can prove these by induction.

Table 1: semi-normal form

(i) $3n$ heaps	
(i-1) each heap has 1 stone	$(C;A;B)$
(i-2) $(3n-1)$ heaps have 1 stone and the other heap has $x(\geq 2)$ stones	$(A;B;C)$
(ii) $(3n+1)$ heaps	
(ii-1) $3n$ heaps have 1 stone and the other heap has $x(\geq 1)$ stones	$(A;B;C)$
(iii) $(3n+2)$ heaps	
(iii-1) each heap has 1 stone	$(B;C;A)$
(iii-2) $(3n+1)$ heaps have 1 stone and the other heap has 2 stones	$(B;A;A)$
(iv) otherwise	$(A;A;A)$

Table 2: semi-reverse form

(i) two heaps	
(i-1) $(1,1)$	$(B;C;A)$
(i-2) $(1,2)$	$(C;A;A)$
(i-3) $(1,x)(x \geq 3)$ or $(2,x)(x \geq 2)$	$(A;A;C)$
(i-4) $(3,3)$	$(A;C;A)$
(ii) three heaps	
(ii-1) $(1,1,1)$	$(C;A;B)$
(ii-2) $(1,1,x)(x \geq 2)$	$(A;B;C)$
(ii-3) $(1,2,x)(x \geq 2)$	$(A;A;C)$
(ii-4) $(2,2,2)$	$(A;C;A)$
(iii) $3n$ heaps $(n \neq 1)$	
(iii-1) each heap has 1 stone	$(C;A;B)$
(iii-2) $(3n-1)$ heaps have 1 stone and the other heap has $x(\geq 2)$ stones	$(A;B;C)$
(iii-3) $(3n-2)$ heaps have 1 stone, one heap has 2 stones and the other heap has $x(\geq 2)$ stones	$(A;A;C)$
(iv) $(3n+1)$ heaps	
(iv-1) $3n$ heaps have 1 stone and the other heap has $x(\geq 1)$ stones	$(A;B;C)$
(iv-2) $(3n-1)$ heaps have 1 stone and the other two heaps have 2 stones	$(A;C;A)$
(v) $(3n+2)$ heaps $(n \neq 0)$	
(v-1) each heap has 1 stone	$(B;C;A)$
(v-2) $(3n+1)$ heaps have 1 stone and the other heap has 2 stones	$(C;A;A)$
(v-3) $(3n+1)$ heaps have 1 stone and the other heap has $x(\geq 3)$ stones	$(A;A;C)$
(v-4) $3n$ heaps have 1 stone, one heap has 2 stones and the other heap has $x(\geq 2)$ stones	$(A;A;C)$
(vi) otherwise	$(A;A;A)$

The winner of reverse form is an open problem. We have the following partial result.

Proposition 2.1 *Consider three heap reverse form games. For all n, m, there is exactly one l such that the last player wins in the game (n, m, l) .*

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Grundy Numbers of Impartial Three Dimensional Chocolate Bar Games

Ryohei Miyadera[1] Yushi Nakaya[2] Masanori Fukui[3] Shunsuke Nakamura[4]

[1,2]Kwansei Gakuin Senior High School [3]Hyogo University of Teacher Education [4]Osaka University

1. Two and three dimensional Chocolate Bar

Chocolate bar games are variants of the CHOMP game in which the goal is to leave your opponent with the single bitter part of the chocolate. The original chocolate bar game consists of a rectangular bar of chocolate with one bitter corner. See Figure 1 for 3×5 rectangular bar. Since the horizontal and vertical grooves are independent, an $m \times n$ rectangular chocolate bar is equivalent to the game of Nim with a heap of $m-1$ stones and a heap of $n-1$ stones. Therefore, the Grundy number of this $m \times n$ rectangular bar is $(m-1) \oplus (n-1)$.

In this paper, we investigate step chocolate bars whose widths are determined by a fixed function of the horizontal distance from the bitter square. When the width of chocolate bar is proportional to the distance from the bitter square and the constant of proportionality is even, the authors have already proved that the Grundy number of this chocolate bar is $(m-1) \oplus (n-1)$, where m is is the largest width of the chocolate and n is the longest horizontal distance from the bitter part. This result was published in [1]. On the other hand, if the constant of proportionality is odd, the Grundy number of this chocolate bar is not $(m-1) \oplus (n-1)$.

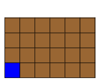
Here, the authors present a necessary and sufficient condition for chocolate bars to have the Grundy number that is equal to $(m-1) \oplus (n-1)$, where m is is the largest width of the chocolate and n is the longest horizontal distance from the bitter part. Let $Z_{\geq 0}$ be the set of non-negative integers.

Definition 1. (i) For any position $\mathbf{p}$ of a game $\mathbf{G}$, there is a set of positions that can be reached by making precisely one move in $\mathbf{G}$, which we will denote by $move(\mathbf{p})$.
(ii) Each position $\mathbf{p}$ of a impartial game has an associated Grundy number $\mathcal{G}(\mathbf{p})$. Grundy number is found recursively: $\mathcal{G}(\mathbf{p}) = mex\{\mathcal{G}(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\}$, where mex is the minimum excluded value.

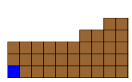
Definition 2. Let f be a monotonically increasing function. For $y, z \in Z_{\geq 0}$ the chocolate bar will consist of $z+1$ columns where the 0th column is the bitter square and the height of the i-th column is $t(i) = \min(f(i), y) + 1$ for i = 0,1,...,z. We will denote this by $CB(f, y, z)$. Thus the height of the i-th column is determined by the value of $\min(f(i), y) + 1$ that is determined by f, i and y.

Definition 3. Each player in turn breaks the bar in a straight line along the grooves into two pieces, and eats the piece without the bitter part. The player who breaks the chocolate bar and eats to leave his opponent with the single bitter block (black block) is the winner.

Example 1. Here are examples of chocolate bar $CB(f, y, z)$.

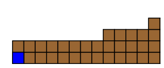
Figure 1. 

$CB(f, 3, 5)$ $f(t) = 4$

Figure 2. 

$CB(f, 3, 12)$

$f(t) = \max(\lfloor \frac{t}{4} \rfloor, 1)$

Figure 3. 

$CB(f, 4, 9)$

$f(t) = \max(\lfloor \frac{t}{2} \rfloor, 2)$

Definition 4. Let h be a monotonically increasing function. h is said to have NS-property if h satisfies the following condition (a).
(a) Suppose that $\lfloor \frac{z}{2^i} \rfloor = \lfloor \frac{z'}{2^i} \rfloor$ for some $z, z' \in Z_{\geq 0}$ and some natural number i. Then we have $\lfloor \frac{h(z)}{2^{i-1}} \rfloor = \lfloor \frac{h(z')}{2^{i-1}} \rfloor$.

Theorem 1. Suppose that h has NS-property in Definition 4. Let $\mathcal{G}_h$ be the Grundy number of $CB(h, y, z)$. Then $\mathcal{G}_h((y, z)) = y \oplus z$, where $y \oplus z$ is the nim-sum (exclusize or) of y and z.

This theorem is a generalization of the main theorem in [1]. The authors are going to talk about the proof of this theorem in the conference.

Definition 5. Let F be a monotonically increasing function. Let $x, y, z \in Z_{\geq 0}$. The three dimensional chocolate bar consists of a set of boxes whose size are $1 \times 1 \times 1$. For $u, v \in Z_{\geq 0}$ such that $u \leq x$, $v \leq y$, the height of column on the position of (u, v) is $\min(F(u, v), y) + 1$. We will denote this by $CB(F, x, y, z)$.

Figure 4.

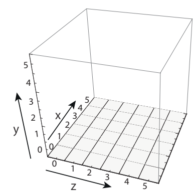
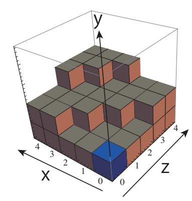


Figure 5.



Theorem 2. Let $F(x, z)$ be a monotonely increasing function. Let $g_n(z) = F(n, z)$ and $h_m(x) = F(x, m)$ for $n, m \in Z_{\geq 0}$. Suppose that g_n and h_m satisfiy NS-property in Definition 4 for any $n, m \in Z_{\geq 0}$. Then $\mathcal{G}_F(\{x, y, z\}) = x \oplus y \oplus z$.

The authors are going to talk about the proof of this theorem in the conference.

2. Chocolate game with a pass move

We consider chocolate game with a pass move. The authors studied this, and the result is in [2]. Here, a postion is said to be $\mathcal{P}$-position if the previous player (the player who will play after the next player) can force a win.

Theorem 3. We consider the disjunctive sum of a chocolate bar $CB(f, y, z)$ to the right of the bitter square and a single strip of chocolate bar of the length of x to the left, where $f(z) = \lfloor (z+s)/k \rfloor$ with an even number k, an odd number s such that $0 < s < k$. We express a position of the game with (x, y, z, p), where $p = 1$ when a pass move is available. Then a position (x, y, z, p) is a $\mathcal{P}$-position if and only if $(x+s) \oplus y \oplus (z+s) \oplus p = 0$ or $(x, y, z, p) = (0, 0, 0, 1)$.

For a proof, see Corollary 4.1 of [2].

Prediction 1. Chocolate bar game $CB(F, x, y, z)$ does not have a simple formula for $\mathcal{P}$-position even if F has NS-property.

This prediction based on the calculation of Mathematica and CGSuite.

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Constant-time testers for generalized shogi, chess, and xiangqi

ITO Hiro*† NAGAO Atsuki* PARK Teagun‡

*School of Informatics and Engineering,
The University of Electro-Communications (UEC); Tokyo, Japan.
†CREST, JST; Tokyo, Japan.
‡Hi-Elecom Co.,Ltd.; Hiroshima, Japan.
*{itohiro, a-nagao}@uec.ac.jp

We present constant-time testing algorithms for the generalized shogi (Japanese chess), chess, and xiangqi (Chinese chess). These problems are known to or believe to be EXPTIME-complete [1, 2]. A testing algorithm (or a tester) for a property accepts an input if it has the property and rejects it if it's far from having the property with high probability (e.g., at least 2/3) by reading only a constant part of the input [3]. A property is said to be testable if there is a tester. Each of the generalized shogi, chess, and xiangqi problem is, given any position on $\sqrt{n} \times \sqrt{n}$ board with $O(n)$ pieces, for testing the property "the player who moves first has a winning strategy." We present that this property is testable for shogi, chess, and xiangqi. The shogi and xiangqi testers are one-sided-error and chess tester is surprisingly no-error! The results on shogi and chess have been already presented by the authors [4]. In this talk, we show them together with the new result on xiangqi. In the last decade, many problems have been found to be testable. However, almost all of such problems are in class NP [3]. It is thought that the series of our results gives the first constant-time testers for EXPTIME-complete problems.

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Polycube Unfoldings Satisfying Conway's Criterion

Stefan Langerman *† Andrew Winslow †

Abstract

We prove that some classes of polycubes admit vertex and edge unfoldings that tile the plane. Surprisingly, every such polycube found has an unfolding satisfying *Conway's criterion* and it remains open whether all polycubes have edge unfoldings satisfying Conway's criterion.

1 Introduction

It is mathematical folklore that the cube can be unfolded to flat by cutting along its edges in 11 non-congruent ways, and that all 11 such unfoldings are non-overlapping *edge unfoldings* tile the plane. Akiyama [1] calls such a polyhedron a *semi-tile-maker* and conjectured that the cube and octahedron are the only two (weakly) convex semi-tile-makers. Shephard [8] introduced the more general class of *tesselation polyhedra* that admit at least one edge unfolding that tiles the plane. They characterize the convex tesselation polyhedra with regular-polygon faces.

Here we mainly consider the existence of tesselation *polycubes*: shapes formed by a union of face-adjacent cubes. Clearly, an edge unfolding is a prerequisite for a tesselation polyhedron. But as with convex polyhedra, it remains open whether every polycube even admits an edge unfolding (Open Problems 21.11 and 22.15 of [5]). The same is true even if the unfolded shape need only be connected by single vertices, so-called *vertex unfoldings*. Prior work on unfolding "gridded" orthogonal polyhedra implies that at least all genus-0 polycubes have vertex unfoldings [4] and a general class of monotone "orthostack" polycubes have edge unfoldings [3].

We obtain three positive results on the existence of polycubes with vertex or edge unfoldings that tile the plane. Curiously, every polycube we find with such an unfolding also has one satisfying Conway's criterion [7], a sufficient condition for tiling the plane that produces tilings with 180° rotational symmetry. Even more curious, the same is true of the tesslation polyhedron found by Akiyama et al. [2]!

*Directeur de recherches du F.R.S.-FNRS.

†Département d'Informatique, Université libre de Bruxelles, ULB CP212, boulevard du Triomphe, 1050 Bruxelles, Belgium, {stefan.langerman,andrew.winslow}@ulb.ac.be

2 Definitions

Polyominoes and polycubes. A *polyomino* is an orthogonal polygon whose edges have identical length; alternatively, a polygon consisting of the edge-connected union of congruent squares. A *polycube* is the three-dimensional equivalent of a polyomino.

Unfoldings. An *edge unfolding* of a polycube is a development of the polycube's surface to a plane as a polyomino whose boundary consists of edges of the polycube. A *vertex unfolding* of a polycube is an edge unfolding where the development must be connected, but not necessarily a polyomino.

Conway's criterion. A polyomino P has a *plane tiling* provided that the plane is the union of (infinitely many) interior-disjoint congruent copies of P. The *boundary word* of a simply connected polyomino P, denoted $\mathcal{B}(P)$, is the circular word over a four-letter alphabet corresponding to directions of the edges as they appear along $\mathcal{B}(P)$. A polyomino P *satisfies Conway's criterion* (and has a plane tiling, see Figure 1) provided $\mathcal{B}(P) = ABC\widehat{A}DE$, where B, C, D, and E are palindromes and $\widehat{A}$ are the opposite directions of A in the reverse order.

3 Results

Theorem 3.1. *Every genus-0 or genus-1 polycube has a vertex unfolding satisfying Conway's criterion.*

A k*-cube* is a polycube consisting of k cubes. The next result is obtained via computer search. Similar search has yielded an unfolding of the 8-cube Dali cross (see Figure 2), answering a question posted by Diaz and O'Rourke [6].

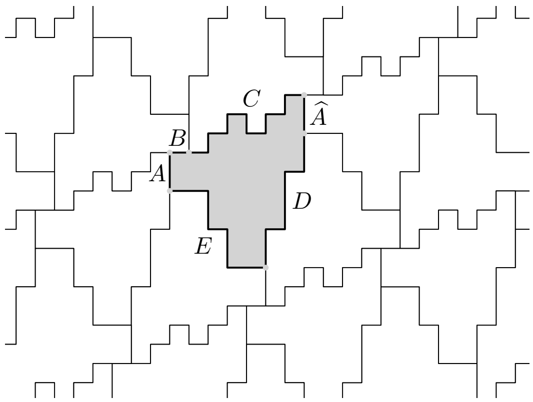

Figure 1: A polyomino satisfying Conway's criterion in the induced plane tiling.

Theorem 3.2. *Every k-cube for $k \leq 7$ has an edge unfolding satisfying Conway's criterion.*

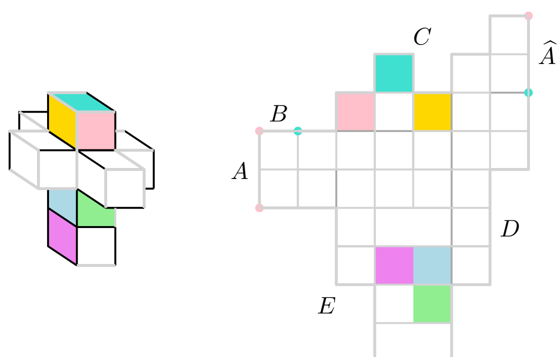

Figure 2: An unfolding of the Dali cross satisfying Conway's criterion. Light and dark gray crease indicate mountain and valley folds, respectively.

The *dual graph* of a polycube has a vertex for each cube and edges between face-adjacent cubes. A *path-like* polycube has a path dual graph. Each path in the dual graph corresponds to a path in the cubic lattice. A *one-layer* polycube has all such paths in a common plane and all length-l paths of an l*-separated* polycube have at most one turn.

Theorem 3.3. *Every path-like one-layer or path-like 4-separated polycube has an edge unfolding satisfying Conway's criterion.*

We can obtain similar results for some classes of tree-like polycubes as well, but do not know how far these results can be extended:

Problem 3.4. *Does every path-like polycube have an edge unfolding satisfying Conway's criterion?*

We lack any example of a polycube without such an unfolding, leading to a strengthened version of Open Problem 22.15 of [5]:

Problem 3.5. *Does every polycube have an edge unfolding satisfying Conway's criterion?*

More specifically, perhaps the *level-1 Menger sponge* is a counterexample: the 20-cube with cube centers at all locations (x, y, z) such that $x, y, z \in \{-1, 0, 1\}$ and at most one of x, y, z is equal to 0.

Problem 3.6. *Does the level-1 Menger sponge polycube have an edge unfolding satisfying Conway's criterion?*

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Continuously Flattening Polyhedra with Two Rigid Adjacent Faces

Jin-ichi Itoh[1] and Chie Nara[2]

[1] Faculty of Education, Kumamoto University, Kumamoto, 860-8555, Japan.*j-itoh@kumamoto-u.ac.jp.*

[2] Meiji Institute for Advanced Study of Mathematical Sciences, Meiji University, Tokyo, 164-8525, Japan. *cnara@jeans.ocn.ne.jp.*

Abstract

We define a book-type polyhedron in three dimensions and show that the surface of any book-type polyhedron satisfying some condition can be flattened continuously to a flat folding such that two adjacent specified faces are rigid, that is, they have no creases during the continuous motion.

We concentrate to fold polyhedral surfaces that are homeomorphic to a sphere. Bern and Hayes [8] proved the existence of a flat folded state of any convex polyhedron by using the disk packing method (see pp. 266-281 in [2]). It was proposed in [2] as an open problem which flat folded state is reached by a continuous motion without cutting the surface and avoiding crossing.

We showed the existence of such continuous motions for Platonic solids by proposing a different method from the disk packing ([3], [5]). For a general convex polyhedron, the authors with C. Vîlcu [4] gave a solution of such problem by using cut loci and the Alexandrov's gluing theorem.

Note that during the continuous flat folding motion for a convex polyhedron, there are *moving* creases, that is, creases on the polyhedron should be changed according to changing its volume, which is followed from the bellow conjecture and Sabitov's theorem ([6, 7]). The creases will cover almost all surface of a given convex polyhedron if we use the cut locus method. Recently, the authors with Z. Abel et al. [1] proved the existence of a continuous motion for any convex polyhedron $\mathcal{P}$ by using the straight skeleton gluing which, for a convex polyhedron, glues points together precisely when there is a ball inside the polyhedron touching those points. In that continuous flat folding motion, there is at least one face of $\mathcal{P}$ which has no crease during the motion. However, for any pair of two adjacent faces of $\mathcal{P}$, at least one of them has (moving) creases.

The purpose of this paper is to find a continuous flat folding motion for a polyhedron $\mathcal{P}$ such that a given pair of two adjacent faces of $\mathcal{P}$ have no creases during the motion, that is, those two faces are rigid. We define a *book-type* polyhedron asfollows.show that there is a continuous flat folding motion for any book-type polyhedron with two given rigid adjacent faces, under some condition.

Definition 1. Let P be a convex n-gon $v_1v_2\cdots v_n$ in the 3-dimensional Euclidean space $\mathbb{R}^3$ with xyz-axes, where $v_1 = (0,0,0)$, $v_i = (x_i, 0, z_i)$ $(1 \leq i \leq n)$ with $x_i \geq 0$ and $0 = z_1 \leq z_2 < z_3 < \cdots < z_{n-1} \leq z_n$, and $v_n = (0,0,z_n)$. P_α denotes the rotated n-gon of P about z-axis with angle $\alpha \geq 0$, and we denote by $v_{(i,\alpha)}$ the rotated point of v_i for $2 \leq i \leq n-1$.

Let $\alpha_0, \alpha_1, \ldots, \alpha_l$ be angles satisfying $\alpha_0 = 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_l < \pi$. Denote by $\mathcal{P} = \mathcal{P}_{\alpha_0,\alpha_1,\ldots,\alpha_l}$ the convex polyhedron obtained as the surface of the convex hull of all vertices of P_{α_i} for $0 \le i \le l$ (see Fig. 1 for an example) and we call $\mathcal{P}$ a *book-type polyhedron*.

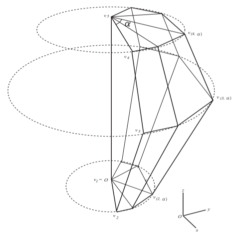

Figure 1: An example of a book-type polyhedron.

We show that there is a continuous flat folding motion for any book-type polyhedron with two given rigid adjacent faces, under some condition first and then for more general cases.

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Note on Area of Moving Creases for Continuous Flattening of Orthogonal Polyhedra

Kazuki Matsubara[1] and Chie Nara[2]

[1] Faculty of Commerce, Chuo Gakuin University,
Chiba, 270-1196, Japan.
additivestructure@gmail.com

[2] Meiji Institute for Advanced Study of Mathematical Sciences,
Meiji University, Tokyo, 164-8525, Japan.
cnara@jeans.ocn.ne.jp.

Abstract

It was proved that any orthogonal polyhedron is continuously flatten by using a property of rhombus. We investigated the method precisely, and found that there are infinitely many ways to flatten them. We prove that the infimum of the area of moving creases is zero. Furthermore, the method can be applied to more general polyhedra.

There are several ways of continuous flattening of convex polyhedra ([1, 4, 5]). However, it is still open question to find a method with small area for moving creases of a given polyhedron, and it is interesting because the area of moving creases should be made by special materials for some products.

A polyhedron is called *orthogonal* if the dihedral angle of each edge is 90° or 270° (see [3]). By an appropriate choice of x, y, z axes for Euclidean space, we can equivalently define a polyhedron to be orthogonal if every face is orthogonal to the x, y, or z axis.

It was proved in [2] that any orthogonal polyhedron is continuously flattened so that all parallel faces to xy-plane are rigid, that is, there are no creases on them. On the other hand, we focus on the area of moving creases for continuous flattening, and prove the following main result.

We show an example in Figure 1.

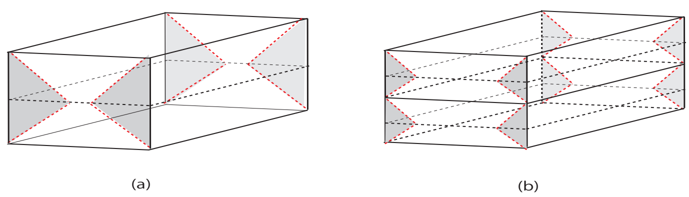

Figure 1: (a) A box with no slice where the region of moving creases are shown in grey triangles, (b) the box with one slice.

Theorem 1. *For any orthogonal polyhedron in the xyz-space, there is a continuous flattening motion whose area of moving creases is extremely small.*

We can also prove similar results for other convex polyhedra, for example, regular tetrahedra.

Theorem 2. *For any regular tetrahedron in the xyz-space, there is a continuous flattening motion whose area of moving creases is extremely small.*

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Continuous Folding Animation of Regular Icosahedron and Truncated Tetrahedron

Jin-ichi Itoh[1], Yuki Kobayashi[2], Chie Nara[3]

[1] Faculty of Education, Kumamoto University, *j-itoh@kumamoto-u.ac.jp*
[2] Department of Architecture and Building Engineering, Tokyo Institute of Technology, *kobayashi.y.bv@m.titech.ac.jp*
[3] Meiji Institute for Advanced Study of Mathematical Sciences, Meiji University, *cnara@jeans.ocn.ne.jp*

Itoh and Nara proved that regular icosahedron is continuously flattened without tearing and stretching by using flat folded rhombuses in [1]. However, there are several ways to fold regular Icosahedron. In this study, we propose the way of flat folding regular icosahedron without self-intersections.

Itoh and Nara proved that truncated tetrahedron is continuously flattened without tearing and stretching by using flat folded rhombuses in [2]. Moreover, every truncated regular polyhedra are continuously flattened without tearing and stretching. Therefore, we made continuous folding animation of truncated tetrahedron at first. Then, we found the existence of some self-intersections in one of the methods. Therefore, we propose the new method of flat folding truncated tetrahedron without self-intersections.

1 Regular Icosahedron

Let P be the regular icosahedron with vertices v_i $(1 \le i \le 12)$. Let h_i $(1 \le i \le 3)$ be midpoints of v_2v_5, v_3v_6, v_1v_4, respectively.

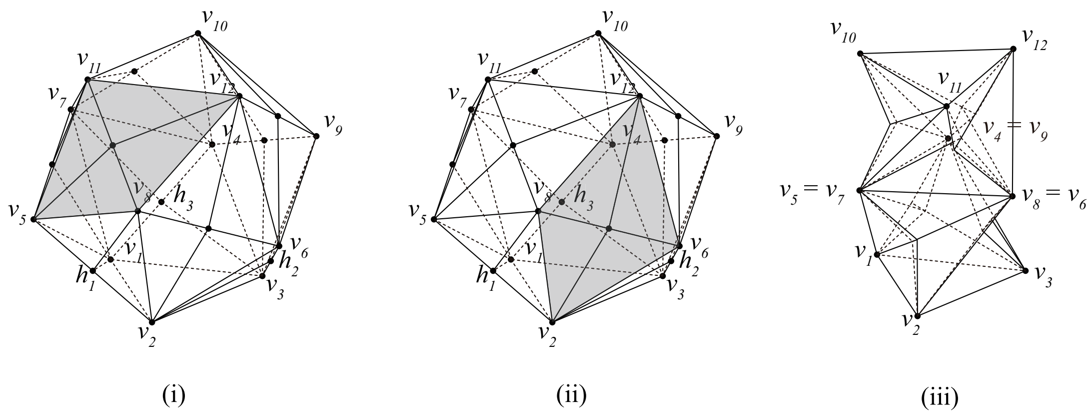

Figure 1: (i) A vertical rhombus. (ii) A horizontal rhombus. (iii) The folded icosahedron whose height is the same value of regular icosahedron.

We consider that the face $\triangle v_1v_2h_1$ is rotated around the edge v_1v_2. Similarly, the face $\triangle v_2v_3h_2$, $\triangle v_1v_3h_3$ are rotated around the edge v_2v_3, v_3v_1, respectively. Let θ be the angle of rotation of $\triangle v_1v_2h_1$. We call the rhombus which is adjacent to the top face $v_{10}v_{11}v_{12}$ or the bottom face $v_1v_2v_3$ with the edge a *horizontal rhombus* (see Fig. 1(i)). We call other rhombus a *vertical rhombus* (see Fig. 1(ii)). We consider the folded icosahedron Q whose height is the same value of the regular icosahedron (Fig. 1(iii)). All vertical rhombuses are flattened on this folded icosahedron. We propose the way of flat folding regular icosahedron as follows.

Step 1: By increasing the height of vertices v_4, v_5, v_6 to the height of v_4^*, v_5^*, v_6^* as θ increases, we fold the icosahedron. In this process, vertical rhombuses are teared. Therefore, we stop this motion just before tearing (see Fig. 1(iii)).

Step 2: We decrease the height of v_4, v_5, v_6 as θ increases (see Fig. 2(ii)). Then, we get the folded icosahedron R. Moreover, there is a continuous folding process of R.

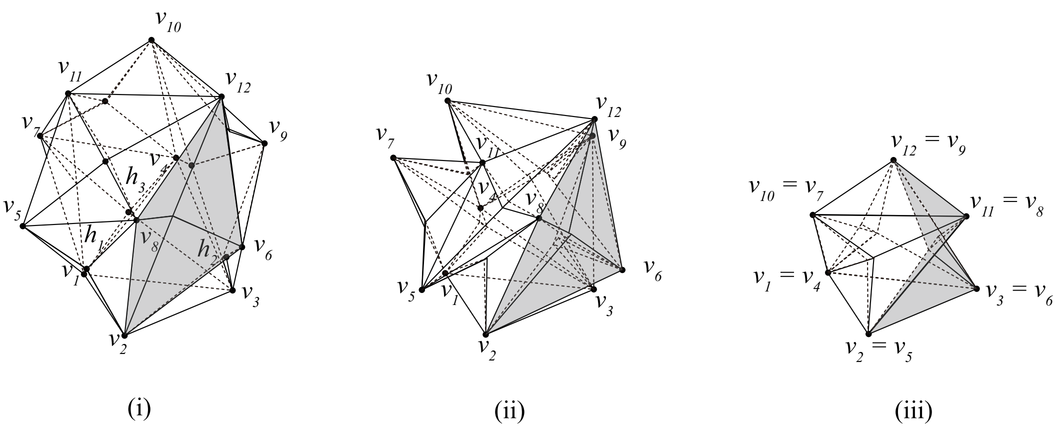

Figure 2: (i) The folded icosahedron just before tearing. (ii) Step 2. (iii) The folded icosahedron R.

2 Truncated Tetrahedron

There are existence of some self-intersections in the one of methods of flat folding truncated tetrahedron in [2] (see Fig. 3(i)). Therefore, we propose the new method of flat folding truncated tetrahedron without self-intersections (see Fig. 3(ii)).

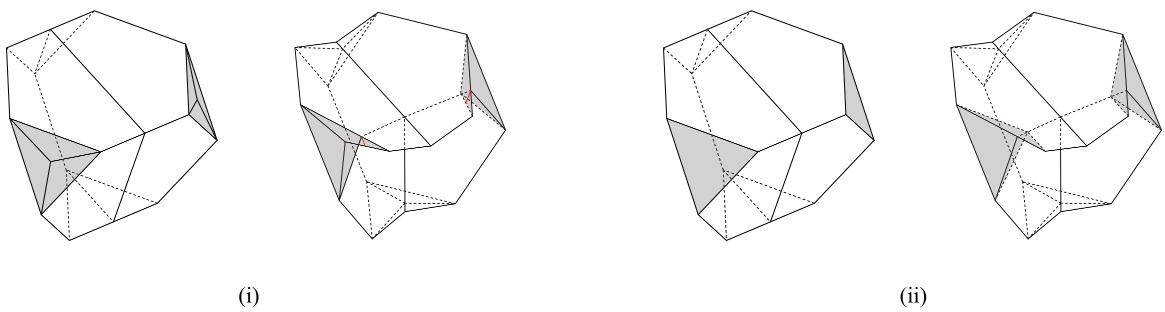

Figure 3: (i) The one of methods of flat folding truncated tetrahedron in [2]. (ii) The new method of flat folding truncated tetrahedron.

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